1. Introduction.

The so called «equilibrium» finite element methods were introduced first by Fraeijs de Veubeke and his school in order to solve problems in linear and non linear elasticity. Their main feature is to provide a continuous stress field, which can be useful in a number of applications. The mathematical analysis has been carried out, for second order linear problems, by Thomas (cfr. [11], [12]) who gave, for the model problem \( \Delta u = f \), sufficient conditions for the convergence and the error bounds in the various cases.

The aim of this paper is to extend the previous analysis to a fourth order model problem, namely

\[
\begin{cases}
\Delta^2 w = p & \text{in } \Omega \\
w = \frac{\partial w}{\partial n} = 0 & \text{on } \partial \Omega.
\end{cases}
\]

\( (P) \)

From the physical point of view this corresponds to the problem of finding the transversal displacement \( w \) of a thin elastic plate which is clamped along the entire boundary and acted by a uniformly distributed load \( p \).
From the mechanical point of view the equilibrium model here presented is essentially a modification of the Hellinger-Reissner principle. It can be viewed as a natural extension of the mixed Hellan-Herrmann-Johnson scheme (see e. g. [4]) where the continuity of the Kirchoff shear stress $K_n$ is also required at the interelement boundaries. This stronger continuity on the stress field allows to reduce the displacement field. In facts, the displacement field, which is a lagrangian multiplier for the constraint $\sigma_{ij,j} = p$, has to be applied only at the interior of each element and in the points where possible discontinuities of the stress field may appear (that is, at the corners). The displacement field will therefore be presented by piecewise linear shape-functions plus possible bubble-functions inside each element. On the other hand, as mentioned before, the stress field will be represented by complete polynomials of degree $k$ in each element, verifying the continuity on $M_n, K_n$.

The mathematical analysis is carried out with a technique quite similar to the one used in [4] for mixed elements. As expected we get optimal $L^2$-error bounds on the stress field, and $O(h^2)$ $L^2$-error bounds on the displacements field.

The layout of the paper is the following: in section 2 the equilibrium formulation of the problem is given; section 3 deals with the numerical approximation; section 4 is devoted to the study of the error bounds and finally in section 5 some numerical results are presented.

For the definition of the functional spaces used in the paper we refer for instance to [7]. The following symbols are used: $|| \cdot ||_{r,A}$ and $| \cdot |_{r,A}$ are the usual norm and semi-norm (cfr. for instance [5]) in the functional spaces $H^r(A)$, $A$ open subset of $\mathbb{R}^2$. $A$ is often omitted when it coincides with the domain $\Omega$.

Finally $| \cdot |$ denotes the usual norm in $L^2(\Omega)$.

The results of this paper have been announced in [10].

2. Equilibrium formulation of the model problem.

From the Hellinger-Reissner principle we get that $(P)$ is equivalent to the following saddle-point problem:

\[ \inf_{\mathbf{w} \in \mathbb{V}} \sup_{\phi \in \mathbb{W}} \left\{ \frac{1}{2} \int_{\Omega} \sum_{ij} \sigma_{ij} \varepsilon^{ij} dx - \tilde{b} \cdot (\mathbf{v}, \phi) + \int_{\Omega} p \phi dx \right\} \]

where

\[ \tilde{\mathbb{V}} = [L^2(\Omega)]^d, \quad \tilde{\mathbb{W}} = H^1_0(\Omega) \]

(4) We denote: $[X]_d = \{ x = (x_i) \mid x_i \in X, x_{i2} = x_{2i}, (i, j = 1, 2) \}$ for any space $X$ of real functions defined on $\Omega$. 