A FINITE ELEMENT TO INTERPOLATE SYMMETRIC TENSORS
WITH DIVERGENCE IN \( L^3 \) \(^{(1)} \)

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ABSTRACT - Inner approximations of the space of second order symmetric tensors with
square-integrable « divergence » over a bounded domain in \( \mathbb{R}^2 \) are built up by means
of a family of affine elements, analogous to the family defined by Raviart and Thomas
in the case of vectors. As an application, we solve the Dirichlet problem for the
biharmonic operator by a method of « equilibrium » type.

Introduction.

For any open set \( A \subset \mathbb{R}^2 \), define the spaces of second order symmetric tensors
\( \mathcal{S}(A) = \{ \tilde{\mathbf{v}} \mid \mathbf{v} = (v_{ij})_{1 \leq i,j \leq 2}, \quad v_{ij} \in L^2(A) \} \) and
\( \mathcal{S}_{L^2}(A) = \{ \mathbf{v} \in \mathcal{S}(A) \mid \text{Div } \mathbf{v} \in L^2(A) \} \)
and

\[ S(A) = \{ \tilde{\mathbf{v}} \in \mathcal{S}(A) \mid \text{Div } \tilde{\mathbf{v}} \in L^2(A) \} \]

where

\[ \text{Div } \tilde{\mathbf{v}} = v_{ijij} \]

is the tensorial divergence of \( \tilde{\mathbf{v}} \). (We denote by \( \cdot/k \) the partial derivative in the
sense of distributions with respect to the variable \( x_k \) (\( k=1,2 \); we omit the sign
of summation over any repeated index).

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In this paper, we construct for any integer \( m \geq 2 \) an affine finite element whose support is any non-degenerate triangle \( K \subseteq \mathbb{R}^2 \), and formed by tensors with polynomial components of degree at most \( m \); by means of it we define an interpolation operator \( \Pi \) on the regular tensors \( \mathbf{v} \in \mathcal{S}(K) \), such that \( \text{Div} \, \Pi \mathbf{v} \) is the orthogonal projection of \( \text{Div} \, \mathbf{v} \) on the space of polynomials of degree at most \( m - 2 \).

For \( m > 2 \), we obtain in a natural way an inner approximation of \( \mathcal{S}(\Omega) \), \( \Omega \subseteq \mathbb{R}^2 \) being a polygonal domain divided into a family \( \mathcal{C} \) of non-degenerate triangles. In effect, a tensor « regular » on each \( K \in \mathcal{C} \) belongs to \( \mathcal{S}(K) \) iff some traces of its satisfy suitable « continuity » conditions on the interelements; the degree of freedom are then chosen in such a way that these conditions can be fulfilled.

However, it is not possible to assign degrees of freedom carrying on each of the traces singly, since some of them are not « invariant » with respect to an affine mapping. What is really « invariant » is a bilinear form which involves globally all the traces we need, and characterizes them in a unique way through its values over a space of test-functions defined on the boundary of the element. Such values are precisely assumed as degrees of freedom.

The element here introduced can be applied in solving boundary value problems for the biharmonic operator via mixed-equilibrium methods: a simple example of such methods is presented in this paper; moreover in Canuto [4] some of the properties of the element are used in proving the convergence of a dual hybrid approximation of a IV-order eigenvalue problem.

We recall that the study of a finite element to interpolate vectors with divergence in \( L^2 \) is due to Raviart - Thomas [6] (see also Thomas [7]).

1. Regularity conditions for symmetric tensors.

Let \( K \subseteq \mathbb{R}^2 \) be a non-degenerate triangle; let us denote by \( a_i \) and \( K'_i \) \((i = 1, 2, 3)\) respectively the vertexes and the sides of \( K \), ordered counterclockwise so that \( a_i \) is common to \( K'_i \) and \( K'_{i+1} \); let \( n = (n_1, n_2) \) be the normal to the boundary \( \partial K \) oriented to the outside, and \( t = (t_1, t_2) \) the tangent oriented counterclockwise. For each \( \mathbf{v} \in \mathcal{S}(K) \) we formally define on the sides of \( K \) the traces

\[
M_n (\mathbf{v}) = v_{ij} n_i n_j
\]

\[
M_{nt} (\mathbf{v}) = v_{ij} n_i t_j
\]

\[
Q_n (\mathbf{v}) = v_{ijj} n_j
\]