ON THE NUMBER OF TRANSITIVE DIGRAPHS
WITH n LABELED VERTICES AND k ARCS (1)

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ABSTRACT - A computer determination of the number $T_{n,k}$ of transitive digraphs on $n$
labeled vertices and $k$ arcs is obtained for $2 \leq n \leq 6$, $0 \leq k \leq n^2 - n$.
Furthermore some formulae are given for determining $T_{n,k}$, $\forall n$ for extremal
values of $k$ (namely, $0 \leq k \leq 6$ and $n^2 - 3n + 4 \leq k \leq n^2 - n$).

Introduction.

The problem of finding the number $T_n$ of transitive digraphs on $n$ labeled
vertices is still an open problem. In [1] the number $T_n$ was determined for
$1 \leq n \leq 7$ and in a recent paper [2] for $1 \leq n \leq 11$; furthermore, in [3], $T_9$ was
separately found.

Aim of this paper is to give a small contribution to this problem of long
standing intractability. More precisely it is devoted to the more general problem
of finding the number $T_{n,k}$ of transitive digraphs with $n$ labeled vertices and
$k$ arcs.

First of all a speedy algorithm is shown for testing the transitivity of a
digraph. Such an algorithm is then utilized for the determination of $T_{n,k}$
for $2 \leq n \leq 6$ and the results are displayed in the corresponding tables; the
table for $n=7$ has been partially filled out. From such tables the behaviour of
$T_{n,k}$ versus $k$ can be deduced.

Finally some formulae are given for determining $T_{n,k}$ for extremal values
of $k$ (namely, $0 \leq k \leq 6$ and $n^2 - 3n + 4 \leq k \leq n^2 - n$). The relevant terminology
used in this paper is the same as that used in ref. [4].

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1. A speedy algorithm for testing the transitivity of a digraph.

First of all we briefly recall two theorems [4] concerning the properties of the successive powers of the adjacency matrix $A(d_n) = \{a_{ij}\}_1^n$ of a digraph $d_n$ on $n$ vertices $v_i (i=1, \ldots, n)$:

a) In $A^m$ the $(i-j)$-entry $a_{ij}^{(m)}$ is the number of sequences in $d_n$ of length $m$ from $v_i$ to $v_j$.

b) For any digraph $d_n$, the diagonal entry $a_{ii}^{(m)}$ of $A^m$ is the number of cycles of length $m$ containing the vertex $v_i$.

Since, by definition, any cycle of length 2 is transitive, we can state $d_n$ to be transitive iff $a_{ij}^{(2)} = 0$ $(i \neq j)$, $\forall a_{ii} = 0$.

Hence the following algorithm is proposed for testing whether $d_n$ is transitive:

0. $a_{ii} = -1, \ \forall i$

1. $\forall (i, j) \exists a_{ij} = 0$

   Set $k = 1$

2. If $a_{ik} < 1$ go to step 3
   
   If $a_{il} = 1$ then stop: $d_n$ is not transitive

3. Increment $k$ by 1

4. If $k < n$ then go to step 2

5. If each $(i, j) \exists a_{ij} = 0$ has been tested then $d_n$ is transitive.

2. Computer determination of $T_{n,k}$ for $2 \leq n \leq 6$.

The previous algorithm has been utilized to count the transitive digraphs with $n$ labeled vertices $(2 \leq n \leq 6)$ and $k$ arcs $(0 \leq k \leq n^2 - n)$. The exhaustive generation of $T_{n,k}$ has been speeded up by making use of a backtrack [5] technique, where it played the role of criterion function.

Such a program has been implemented in FORTRAN language and the values $T_{n,k}$ have been computed on the IBM 370/168 computer of the PTT Ministry Data Processing Center.

The obtained results are displayed in the following tables. Since the complete enumeration of $T_{7,k}$ had appeared too heavy, the corresponding table has been only partially filled out.