A Note on the $\beta$-Measure for Digraph Competitions

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Abstract
Digraph games are cooperative TU-games associated to digraph competitions: domination structures that can be modeled by directed graphs. Examples come from sports competitions or from simple majority win digraphs corresponding to preference profiles for a group of individuals within the framework of social choice theory. Brink and Gilles (2000) defined the $\beta$-measure of a digraph competition as the Shapley value of the corresponding digraph game. This paper provides a new characterization of the $\beta$-measure.

Key Words: Digraph competition, $\beta$-measure.

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1 Introduction

A directed graph can represent various domination structures that are based on (partial) pairwise comparisons. An obvious example is a sports competition in which several teams play matches against each other and the digraph summarizes the results of the various matches. Other examples include the results of paired comparison experiments for example within a group of alternative medicines, the results of aggregated pairwise preferences of a group of individuals, based e.g. on simple majority voting, over a certain set of alternatives, or, within a completely different framework, the hierarchical structure in economic organizations.

In the sequel we use the term digraph competition for a domination structure modeled by a digraph. The main issue under consideration is how to measure the “strength” of each node in a digraph competition. We continue on the lines set out by Van den Brink and Borm (2002) and consider an arbitrary digraph competition as a special type of allocation problem where we initially assume that each node is assigned equal weight (say equal to one). Measuring strength then can be seen as “fairly” reallocating these weights taking into account the domination structure that is represented by the digraph. For this aim a digraph game is associated to each digraph competition: in this game the players correspond to the nodes and the value of a coalition represents the maximal total weight for which there is no rightful direct claim from outside this coalition. Here a player has a rightful claim on the weights of all nodes that he dominates directly. One way to measure the strength of the nodes in a digraph competition is to consider the Shapley value of the associated digraph game (cf. Shapley (1953)). We follow Van den Brink and Gilles (2000) and Van den Brink and Borm (2002) in calling this the $\beta$-measure of the underlying digraph competition.

Van den Brink and Borm (2002) extensively analyze the class of digraph games. Among other things it turns out that these games are convex and that the Shapley value of these games, -by convexity the barycentre of the core-, is also the average of rather intuitive so-called simple score vectors (which are in the core) related to specific sub-digraphs. Borm et al. (2002) exploit the possibility to start out from arbitrary initial weights by investigating the limit behavior in the iterative process that takes the Shapley value of the digraph game as new weight parameters of the nodes to determine the new digraph game in the next step. The resulting limit measure is called the $\lambda$-measure. In Borm et al. (2004) the particular application area of aggregate preferences over alternatives is investigated: the $\beta$- and $\lambda$-measures form the basis for defining and analyzing two new social choice correspondences.

The current short note provides a characterization of the $\beta$-measure on the class of all digraph competitions using the properties of component efficiency, symmetry, triviality and decomposition additivity. Characterizations of the $\beta$-measure on specific subclasses of digraph competitions are provided in Van den Brink (1994).

The paper is organized as follows. Section 2 recalls the main definitions