ON A PURSUIT GAME ON CAYLEY GRAPHS

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The game cops and robbers is considered on Cayley graphs of abelian groups. It is proved that if the graph has degree \( d \), then \( [(d+1)/2] \) cops are sufficient to catch one robber. This bound is often best possible.

1. Introduction

We consider the following game, called cops and robbers. There is a finite, connected, undirected graph \( G=(V,E) \), \( m \) cops and one robber. First the cops choose one vertex each as initial position. Next the robber makes his choice. Afterwards they move alternately along the edges of the graph or stay. (First the cops, then the robber.) The game is with full information, that is, everyone knows the position of the others.

Denote by \( c(G) \) the minimum value of \( m \) for which \( m \) cops have a winning strategy, i.e., they have an algorithm to catch the robber (at least one cop gets to the same vertex as the robber) no matter how he plays.

This game was studied by several authors: Aigner and Fromme [1], Andreae [2], [3], Hamidoune [5], Quilliot [9], [10], Maamoun and Meyniel [7].

For a finite group \( H \) and a subset \( S \) of its elements satisfying \( S = S^{-1} \) one defines the Cayley graph \( C(H, S) = (H,E) \) with vertex set \( H \) and edge set \( E = \{\{h;hs\}:h\in H, s \in S\} \). Let \( H_0 = \langle S \rangle \) be the subgroup of \( H \) generated by \( S \). Then \( C(H, S) \) is connected if and only if \( H = H_0 \). Otherwise it is the disjoint union of \( |H:H_0| \) copies of the connected Cayley graph \( C(H_0, S) \).

The main result of this paper is the following:

**Theorem 1.** Suppose that \( C(H, S) \) is a connected Cayley graph of the abelian group \( H \). Then we have

\[
c(C(H, S)) \leq [(|S| + 1)/2].
\]

Inequality (1.1) improves the bound \([3|S|/4]\), which was obtained by Hamidoune [5], whose ideas are used in the proof of (1.1).
Let us note also that the assumption on the commutativity of \( H \) cannot be dropped. Using the constructions of Margulis [8] and Imrich [6] for Cayley graphs with large girth, it is shown in [4] that there exist Cayley graphs \( G \) of degree \( d \), for every \( d \geq 3 \), and with \( c(G) \) arbitrarily large. Let \( C_n \) denote the cycle of length \( n \) and let \( \delta(G) \) denote the minimum degree of \( G \).

Aigner and Fromme [1] showed that if \( G \) does not contain \( C_3 \) and \( C_4 \) then \( c(G) \geq \delta(G) \) holds.

Their proof can be adapted to show the following:

**Proposition 2.** Suppose that in \( G \) any two vertices are connected by at most 2 paths of length at most 2. Then \( c(G) \equiv \delta(G)/2 \). Moreover, if \( G \) contains no \( C_3 \), then \( c(G) \equiv \delta(G)/2 \) holds.

This proposition can be used to construct many Cayley graphs \( G \) with \( c(G) \equiv \delta(G)/2 \), i.e., giving equality in Theorem 1. In particular, if \( S \) is a minimal generating set of the abelian group \( H \), then Theorem 1 and Proposition 2 imply \( c(H, S) = \lfloor (|S| + 1)/2 \rfloor \).

2. The proof of Theorem 1

To prove the theorem let us introduce the following restricted version of the game cops and robbers on \( C(H, S) \). Let \( T \) be an arbitrary subset of \( S \) and suppose that from vertex \( h \) the robber can move only to one of the vertices \( \{ht : t \in T\} \), or stay in \( h \).

That is, the robber can only use the generators in \( T \) while the cops those of \( S \).

Let \( c(H, S, T) \) denote the minimum number of cops needed to catch the robber in this restricted game, where \( H = \langle s \rangle \) is assumed.

**Theorem 3.** For every finite abelian group \( H \) and all subsets \( T \subsetneq S \subsetneq H \) one has \( c(H, S, T) \equiv \lfloor (|T| + 1)/2 \rfloor \).

Note that Theorem 1 is the special case \( S = T \).

**Proof of Theorem 3.** We prove Theorem 3 by double induction: we apply induction on \( |H| \) and for fixed \( H \) on \( |T| \).

The case \( |T| = 1 \) is trivial: the one and only cop moves in a finite number of steps to the connected component of \( C(H, T) \) where the robber is. Note that the robber cannot leave this component. If \( T = \emptyset \), then the robber is caught. If \( T = \{t\} \), then the cop keeps using \( t^{-1} \) and catches the robber in less than the order of \( t \) steps.

Also, if \( |T| = 2 \), \( T = \{t, t^{-1}\} \), then \( c(H, S, T) \equiv \lfloor (2 + 1)/2 \rfloor = 2 \) follows easily. Both cops move to the connected component of \( C(H, T) \) containing the robber. Then one of them keeps using \( t \), the other \( t^{-1} \) and they catch the robber in less than half of the order of \( t \) moves.

From now on we may assume that there are two elements \( s, t \in T \) such that \( st \neq 1 \).

Let \( K = \langle st \rangle \) be the group generated by \( st \).