EXPANDING GRAPHS CONTAIN ALL SMALL TREES

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The assertion of the title is formulated and proved. The result is then used to construct graphs with a linear number of edges that, even after the deletion of almost all of their edges or almost all of their vertices, continue to contain all small trees.

1. Introduction

If $H$ is an undirected graph, $V(H)$ will denote its set of vertices and $E(H)$ will denote its set of edges. If $X \subseteq V(H)$, $\Gamma_H(X)$ will denote the set of neighbors in $H$ of vertices in $X$. If $X$ is a set, $|X|$ will denote its cardinality.

The following theorem, which is implicit in a result of Pósa [5], has been given an elegant proof by Lovász ([3], Ch. 10. Problem 20).

**Theorem 0.** If $H$ is a non-empty graph such that, for each $X \subseteq V(H)$ with $|X| \leq n$,

$$|\Gamma_H(X) \setminus X| \geq 2|X| - 1,$$

then $H$ contains a path with $3n - 2$ vertices.

Using Theorem 0, Beck [2] proved an upper bound of the form $O(n)$ for the minimum possible number of edges in graphs that, even after the deletion of half their edges, continue to contain a path with $n$ vertices; Alon and Chung [1] have given an explicit construction for such graphs.

The main result of this paper is the following theorem.

**Theorem 1.** If $H$ is a non-empty graph such that, for every $X \subseteq V(H)$ with $|X| \leq 2n - 2$,

$$|\Gamma_H(X)| \geq (d + 1)|X|,$$

then $H$ contains every tree with $n$ vertices and maximum degree at most $d$.

Since a path with $n$ vertices is the unique tree with $n$ vertices and maximum degree 2, Theorem 1 generalizes the essence of Theorem 0 from paths to trees. Using Theorem 1, the arguments of Beck [2] and Alon and Chung [1], and a recent result of Lubotzky, Phillips and Sarnak [4], we prove the following theorem.

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Theorem 2. Let $\delta > 0$ and $d$ be fixed. For every $n$ there is a graph $F$ with $O(n)$ edges that, even after deletion of all but $\delta |E(F)|$ edges, continues to contain every tree with $n$ vertices and maximum degree at most $d$.

For $\delta = 1/2$, Beck [2] proved an upper bound, without an explicit construction, of the form $O(n(\log n)^2)$. We shall also prove the following theorem.

Theorem 3. Let $\varepsilon > 0$ and $d$ be fixed. For every $n$ there is a graph $F$ with $O(n)$ edges that, even after deletion of all but $\varepsilon |V(F)|$ vertices, continues to contain every tree with $n$ vertices and maximum degree at most $d$.

2. Proof of Theorem 1

If $T$ is a tree and $H$ is a graph, a map $f: V(T) \rightarrow V(H)$ will be called an embedding of $T$ in $H$ if it is injective and $f(v)$ and $f(w)$ are adjacent in $H$ whenever $v$ and $w$ are adjacent in $T$. A tree $T$ will be called small if it has at most $n$ vertices and maximum degree at most $d$. (The parameters $n$ and $d$ will remain fixed throughout this proof.) A graph $H$ will be called expanding if, for every $X \subseteq V(H)$ with $|X| \leq 2n-2$,

$$|\Gamma_H(X)| \geq (d+1)|X|.$$ 

Our goal is to show that if $T$ is small and $H$ is non-empty and expanding, then there is an embedding of $T$ in $H$. To achieve this we shall define a class of "good" embeddings. We shall then show that this class has the following two properties.

Property 1. If $T$ consists of a single vertex and $H$ is a non-empty expanding graph, then there is a good embedding of $T$ in $H$.

Property 2. If $T$ is a small tree and $S$ is a subtree of $T$ obtained by deleting a leaf and the edge incident with it, then any good embedding of $S$ in an expanding graph $H$ can be extended to a good embedding of $T$ in $H$.

When this has been done, it will follow by induction on $|T|$ that, if $T$ is a small tree and $H$ is a non-empty expanding graph, then there is a good embedding of $T$ in $H$. If $|V(T)| = 1$, this follows from Property 1. If $|V(T)| \geq 2$, let $S$ be any tree obtained from $T$ by deleting a leaf and the edge incident with it. By inductive hypothesis, there is a good embedding of $S$ in $H$, and by Property 2, this can be extended to a good embedding of $T$ in $H$. This completes the induction and the proof of Theorem 1.

To define good embeddings, we shall need some auxiliary definitions. Let $f$ be an embedding of a tree $T$ in a graph $H$. If $X \subseteq V(H)$, we shall define the assets $A_f(X)$ of $X$ under $f$ to be $|\Gamma_H(X) \setminus f(V(T))|$. If $x \in V(H)$, we shall let $J_f(x)$ denote the degree of $f^{-1}(x)$ in $T$ if $x \notin f(T)$, and 0 otherwise. We shall let $B_f(x)$ denote $d - J_f(x)$. If $X \subseteq V(H)$, we shall define the liabilities $B_f(X)$ of $X$ under $f$ to be $\sum_{x \in X} B_f(x)$, and the balance $C_f(X)$ of $X$ under $f$ to be $A_f(X) - B_f(X)$. A set $X \subseteq V(H)$ will be called solvent under $f$ if $C_f(X) \geq 0$, critical under $f$ if $C_f(X) = 0$, and bankrupt under $f$ if $C_f(X) < 0$. Finally, we arrive at the key definition. An embedding $f$ of a tree $T$ in a graph $H$ is good if every $X \subseteq V(H)$ with $|X| \leq 2n-2$ is solvent. It remains to prove Properties 1 and 2.