PACKING NEARLY-DISJOINT SETS

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De Bruijn and Erdős proved that if $A_1, ..., A_k$ are distinct subsets of a set of cardinality $n$, and $|A_i \cap A_j| \leq 1$ for $1 \leq i < j \leq k$, and $k > n$, then some two of $A_1, ..., A_k$ have empty intersection. We prove a strengthening, that at least $\frac{1}{n}$ of $A_1, ..., A_k$ are pairwise disjoint. This is motivated by a well-known conjecture of Erdős, Faber and Lovász of which it is a corollary.

1. Introduction

Throughout this paper we shall only be concerned with finite structures, and further reference to this will be omitted.

If $\mathcal{A}$ is a collection of sets, we say that $\mathcal{A}$ is nearly-disjoint if $|A \cap A'| \leq 1$ for distinct $A, A' \in \mathcal{A}$. We begin by describing a conjecture of Erdős, Faber and Lovász (see for example [2]), which motivated this paper. It is the following.

(1.1) (Conjecture). Let $n > 0$ be an integer, and let $\mathcal{A}$ be a nearly-disjoint collection of sets, each with cardinality $n$, and let $|\mathcal{A}| = n$. Then the elements of $\bigcup (A : A \in \mathcal{A})$ may be coloured with $n$ colours, so that for each $A \in \mathcal{A}$, all the elements of $A$ receive different colours.

It may be shown to be equivalent to the following.

(1.2) (Conjecture). Let $\mathcal{A}$ be a nearly-disjoint collection of subsets of a set of cardinality $n > 0$. Then the members of $\mathcal{A}$ may be coloured with $n$ colours, so that distinct $A, A' \in \mathcal{A}$ receive different colours if $A \cap A' \neq \emptyset$.

We prove this equivalence in section 2. Our main result is the following.

(1.3) Let $\mathcal{A}$ be a nearly-disjoint collection of subsets of a set of cardinality $n > 0$. Then there exist $A_1, ..., A_p \in \mathcal{A}$, pairwise disjoint, where $p \geq \frac{1}{n} |\mathcal{A}|$.
Clearly this is a corollary of (1.2) since some colour class must contain at least 
\[ \frac{1}{n} |\mathcal{A}| \] members of \( \mathcal{A} \). We prove (1.3) in section 3, and in section 4 we find the collections 
for which 1.3 is best possible. Section 5 contains some concluding remarks.

Let us mention in passing that (1.3) contains a well-known result of De Bruijn 
and Erdős [1], the following.

(1.4) Let \( \mathcal{A} \) be a collection of subsets of a set of cardinality \( n > 0 \), such that \( |A \cap A'| = 1 \) 
for distinct \( A, A' \in \mathcal{A} \). Then \( |\mathcal{A}| \leq n \).

2. The Equivalence of (1.1) and (1.2)

Let \( \mathcal{A} \) be a nearly-disjoint collection of subsets of a set \( E \). (Note that “collection” 
means “set”, and a collection has no repeated members.) Consider the following 
statements about \( \mathcal{A} \).

(2.1) \( |\mathcal{A}| = n > 0 \).
(2.2) \( |A| = n \) for each \( A \in \mathcal{A} \).
(2.3) The elements of \( E \) may be coloured with \( n \) colours so that for each \( A \in \mathcal{A} \), all the 
elements of \( A \) receive different colours.

Thus conjecture (1.1) asserts that (2.1) and (2.2) imply (2.3). Now consider the 
statement

(2.4) \( |A| \leq n \) for each \( A \in \mathcal{A} \).

(1.1) is equivalent to the assertion that (2.1) and (2.4) imply (2.3), for if some 
\( A \in \mathcal{A} \) has \( |A| < n \), we simply add \( n - |A| \) new elements to \( A \). Finally, consider the 
following statement.

(2.5) For distinct \( x, x' \in E \), there exists \( A \in \mathcal{A} \) such that \( |A \cap \{x, x'\}| = 1 \).

We may add (2.5) to our list of hypotheses; for if (2.1), (2.4) and (2.5) together 
imply (2.3), then (2.1) and (2.4) imply (2.3), as is easily seen. But (2.4) is implied by 
(2.1) and (2.5). Hence (1.1) is equivalent to the assertion that (2.1) and (2.5) imply 
(2.3). However, (2.5) ensures that the set-element dual of \( \mathcal{A} \) is another collection of 
sets, and it is clearly nearly-disjoint. (1.2) is merely the assertion that (2.1) and (2.5) 
imply (2.3), expressed in terms of the set-element dual of \( \mathcal{A} \). Thus (1.1) and (1.2) are 
equivalent.

3. Proof of the theorem 1.3

Let \( p \) be the least integer with \( p \geq \frac{1}{n} |\mathcal{A}| \). We proceed by induction on \( p \).
The result is trivial if \( p = 0 \), and so we assume that \( p > 0 \) and hence that \( \mathcal{A} \neq \emptyset \).

(3.1) If \( A \in \mathcal{A} \) and \( |A| = m \), then there are at least \((p-2)m + n + 1\) members of \( \mathcal{A} \) which have non-empty intersection with \( A \).

Proof. If there are more than \((n-m)(p-2)\) members of \( \mathcal{A} \) disjoint from \( A \), then by 
induction some \( p-1 \) of them are pairwise disjoint, and these together with \( A \) give the