THE DIAMETER OF RANDOM REGULAR GRAPHS

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We give asymptotic upper and lower bounds for the diameter of almost every \( r \)-regular graph on \( n \) vertices \( (n \to \infty) \).

Though random graphs of various types have been investigated extensively over the last twenty years, random regular graphs have hardly been studied. The reason for this is that until recently there was no formula for the asymptotic number of labelled \( r \)-regular graphs of order \( n \). Such a formula was given by Bender and Canfield [1]. Even more recently one of the present authors [3] gave a simpler proof of the same formula. More importantly, [3] contains a model for the set of regular graphs which can be used to study labelled random regular graphs. Our aim is to investigate the diameter: we shall show that for a fixed \( r \) most \( r \)-regular graphs of order \( n \) have about the same diameter. Our results have some bearing on certain extremal problems concerning graphs of small diameter and small maximum degree (see [2, Ch. IV]). For results about the diameter of the customary random graphs see [4], [5] and [6].

Let us start with the model mentioned above. Let \( r \geq 3 \) be fixed and denote by \( G(n, r\text{-reg}) \) the probability space of all \( r \)-regular graphs with a fixed set of \( n \) labelled vertices. Here we assume that \( rn \) is even and any two graphs have the same probability. We shall say that almost every (a.e.) \( r \)-regular graph has a certain property if the probability that a member of \( G(n, r\text{-reg}) \) has this property tends to 1 as \( n \to \infty \). Let \( W_1, W_2, \ldots, W_n \) be disjoint \( r \)-element sets. A configuration with vertex set \( W = \bigcup_{i=1}^{n} W_i \) is a partition of \( W \) into pairs. We call these pairs the edges of the configuration and we denote by \( \Omega \) the set of all configurations with vertex set \( W \). Once again we view \( \Omega \) as a probability space in which all points are equiprobable. Given a configuration \( F \in \Omega \) we may try to construct an \( r \)-regular graph \( \varphi(F) \) with vertex set \( \{W_1, W_2, \ldots, W_n\} \) as follows. Join two vertices \( W_i \) and \( W_j \) by an edge iff the configuration \( F \) contains an edge having one vertex in \( W_i \) and the other in \( W_j \). Clearly \( \varphi(F) \) is an \( r \)-regular graph if \( F \) has no edge joining two vertices of the same class \( W_i \), nor has it two edges joining vertices in the same two classes \( W_i \) and \( W_j \). All we need

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from [3] is that the probability that a configuration $F$ has these two properties is bounded away from 0. More precisely, we need the following immediate consequence of this assertion: if a.e. configuration has a certain property then a.e. $r$-regular graph has the corresponding property. The property we are concerned with is that of having a certain diameter.

Let $F$ be a configuration and $W_i, W_j$ arbitrary classes. Define the distance between $W_i$ and $W_j$ as the minimal $k$ for which one can find classes $W_i_0 = W_i, W_i_1, W_i_2, \ldots, W_i_k = W_j$, such that for every $j$, $0 \leq j < k$, the class $W_i_j$ is joined to $W_i_{j+1}$ by an edge of $F$. If there is no such $k$ then the distance between $W_i$ and $W_j$ is said to be infinite. The distance between $W_i$ and $W_j$ is denoted by $d(W_i, W_j)$ or $d_r(W_i, W_j)$. Note that $d(W_i, W_j) = 0$ if and only if $W_i = W_j$. The diameter of $F$ is the maximal distance between two classes. From what was said earlier it is clear that if a.e. configuration has diameter at least $d'(n)$ then a.e. $r$-regular graph has diameter at least $d'(n)$, and if a.e. configuration has diameter at most $d''(n)$ then a.e. $r$-regular graph has diameter at most $d''(n)$. Thus our aim is to give bounds for the diameter of a.e. configuration.

In our proofs we shall often find it convenient to work with another description of $\Omega$, namely we shall find it convenient to select the edges of a random configuration one by one, taking those nearest to a fixed class $W_i$ first. To be precise, we shall construct inductively sets $E_i, S_i$ and $M_i$ in such a way that $S_i$ will be the set of indices $j$ with $d(W_i, W_j) = i$ in the final configuration (i.e. $S_i$ will be the set of indices of the classes $W_j$ on the sphere of radius $i$ about $W_i$), $E_i$ will be the set of edges incident with the classes at distance less than $i$ from $W_i$ and $M_i$ will be the set of vertices (members of the classes $W_j$) not incident with any edge in $E_i$. Thus $E_{i+1}$ is obtained from $E_i$ by adding to $E_i$ all the edges incident with the vertices in $L_i = M_i \cap \bigcup_{j \in S_i} W_j$.

Set $E_0 = \emptyset$, $S_0 = \{1\}$ and $M_0 = \bigcup_{i=2}^{n} W_i$. Suppose we have defined $E_i, S_i$ and $M_i$. In order to define $E_{i+1}$, $S_{i+1}$ and $M_{i+1}$, we pass through a number of intermediate stages, corresponding to the selection of single edges.

Set $E_i(0) = E_i$, $S_i(0) = \emptyset$, $M_i(0) = M_i$ and $L_i(0) = L_i = M_i \cap \bigcup_{i \in S_i} W_i$. Suppose we have defined $E_i(j)$, $S_i(j)$, $M_i(j)$ and $L_i(j)$. If $L_i(j) = \emptyset$, put $E_{i+1} = E_i(j)$, $S_{i+1} = S_i(j)$, $M_{i+1} = M_i(j)$ and $L_{i+1} = M_{i+1} \cap \bigcup_{j \in S_i} W_i$. Otherwise pick a vertex $x \in L_i(j)$ (say take the first vertex in $L_i(j)$ in some predefined order), give all vertices of $M_{i+1} - \{x\}$ the same probability and choose one of them, say $y$. Set $E_i(j+1) = E_i(j) \cup \{(x, y)\}$, $L_i(j+1) = L_i(j) - \{x, y\}$ and $M_i(j+1) = M_i(j) - \{x, y\}$. Finally, if $y \in W_i$ and $i \in S_i$, set $S_i(j+1) = S_i(j) \cup \{i\}$, and otherwise put $S_i(j+1) = S_i(j)$. Since $M_i(j+1)$ has two fewer vertices than $M_i(j)$, the process does terminate so it does define $E_{i+1}$, $S_{i+1}$ and $M_{i+1}$.

After a certain number of steps we must arrive at $E_k = \emptyset$ and so $S_k = \emptyset$. Then the set $E_k$ is exactly the edge set of the component containing $W_i$ in a random configuration. If $M_k = \emptyset$, we have constructed the whole configuration. However, even if we end up with $M_k \neq \emptyset$, we have constructed the part of a random configuration nearest to $W_i$ in the following sense: for every $i \leq k$ the probability of having $E_i = \emptyset$ is the same as the probability that $E$ is the set of edges incident with the classes at distance less than $i$ from $W_i$. (In fact, even if $M_k \neq \emptyset$, we can construct an entire random configuration by iterating the process above, but this will not concern us.)

Now we are well prepared to prove our first theorem.