Let $V_{\text{fin}}$ and $E_{\text{fin}}$, resp. denote the classes of graphs $G$ with the property that no matter how we label the vertices (edges, resp.) of $G$ by members of a linearly ordered set, there will exist paths of arbitrary finite lengths with monotonically increasing labels. The classes $V_{\text{inf}}$ and $E_{\text{inf}}$ are defined similarly by requiring the existence of an infinite path with increasing labels. We prove $E_{\text{inf}} \subseteq V_{\text{inf}} \subseteq V_{\text{fin}} \subseteq E_{\text{fin}}$. Finally we consider labellings by positive integers and characterize the class corresponding to $V_{\text{fin}}$.

1. Introduction

Let $\mathcal{D}$ be a digraph of finite chromatic number $k$. In [2] Gallai proved that $\mathcal{D}$ contains an oriented path of length $k-1$. Denote by $\mathcal{K}_{\text{fin}}$ the class of graphs $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ with the following property: For every linearly ordered set $\mathcal{L}$ and for every injective mapping $f : V(\mathcal{G}) \to \mathcal{L}$ there exists an arbitrarily long finite simple path $\{x_1, x_2\}, \ldots, \{x_{n-1}, x_n\}$ such that $\{x_i, x_{i+1}\} \in E(\mathcal{G})$ and $f(x_i) < f(x_{i+1})$ for all $i$ and $n$, $1 \leq i \leq n-1$. It follows easily from Gallai's theorem that $\mathcal{G} \in \mathcal{K}_{\text{fin}}$ iff $\mathcal{G} = \mathcal{G}$.

In [3] and [5] edge weighted graphs were investigated. It was proved in [3], [5] that any graph $\mathcal{G}$ with minimum degree $\geq p$ has the property that for every injection $g : E(\mathcal{G}) \to \mathcal{L}$ ($\mathcal{L}$ is a linearly ordered set) there is a simple path $\{x_1, x_2\}, \ldots, \{x_{r-1}, x_r\}$ where $r = \left\lfloor \frac{1}{2} + \sqrt{\frac{1}{4} + p} \right\rfloor$, such that $g(\{x_1, x_2\}) \leq g(\{x_2, x_3\}) \leq \ldots \leq \ldots \leq g(\{x_{r-1}, x_r\})$.

Using these facts one can show that the class $\mathcal{K}_{\text{fin}}$ is a proper subclass of the class $\mathcal{K}_{\text{fin}}$ of all graphs which contain arbitrarily long monotone paths with respect to any edge ordering. Indeed, if $\mathcal{G} \in \mathcal{K}_{\text{fin}}$ then $\mathcal{G} = \mathcal{G}$ and, according to a theorem of Erdős—de Bruijn [1], $\mathcal{G}$ contains finite subgraphs of arbitrarily large chromatic number and thus with large degrees. To show $\mathcal{K}_{\text{fin}} \neq \mathcal{K}_{\text{fin}}$ consider bipartite graphs with large degrees. We show here that the above observation cannot be extended to graphs containing infinite monotone paths: The class $\mathcal{K}_{\text{fin}}$ of all graphs containing an
infinite monotone path for any ordering of edges is a proper subclass of the class $\mathcal{V}_{\text{fin}}$ of all graphs containing an infinite path for any valuation of vertices.

Thus

$$\mathcal{E}_{\text{inf}} \subseteq \mathcal{V}_{\text{inf}} \subseteq \mathcal{V}_{\text{fin}} \subseteq \mathcal{E}_{\text{fin}}.$$ 

In the last section we investigate those countable graphs $G$ which contain an infinite monotone path for any ordering of vertices by positive integers. We show that the obvious sufficient condition—there exists a subgraph $H \subseteq G$ such that all degrees of $H$ are infinite—is also necessary for the existence of an infinite monotone path for every such ordering.

2. Basic notions

The graphs considered here are simple, undirected and without loops. Let $\mathcal{G}$ be a graph. Then $V(\mathcal{G})$ and $E(\mathcal{G})$ denote the vertex and edge sets of $\mathcal{G}$, respectively. A mapping $\varphi: V(\mathcal{G}) \rightarrow V(\mathcal{G}')$ is said to be a homomorphism from a graph $\mathcal{G}$ to a graph $\mathcal{G}'$ if $\{x, y\} \in E(\mathcal{G})$ implies $\{\varphi(x), \varphi(y)\} \in E(\mathcal{G}')$. By a path of length $k$ (infinite path) we understand a sequence $(e_1, e_2, ..., e_k)$ (infinite sequence $(e_1, e_2, ...)$) of distinct edges such that $e_i$ and $e_j$ have a common vertex iff $|i - j| \leq 1$. If convenient we identify a path with its set of vertices.

By a digraph $\mathcal{D}$ we understand a pair $(V, A)$ where $V = V(\mathcal{D})$ is the set of vertices and $A = A(\mathcal{D})$ is the set of arcs i.e. ordered pairs of distinct vertices. A digraph $\mathcal{D} = (V(\mathcal{D}), A(\mathcal{D}))$ is called an orientation of a graph $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ if $V(\mathcal{D}) = V(\mathcal{G})$, $A(\mathcal{D}) \cup A(\mathcal{D})^{-1} = E(\mathcal{G})$ and $A(\mathcal{D}) \cap A^{-1}(\mathcal{D}) = \emptyset$. By a path of length $k$ (infinite path) in $\mathcal{D}$ we understand a sequence $(a_1, a_2, ..., a_k)$ (an infinite sequence $(a_1, a_2, ...)$) of distinct arcs of $\mathcal{G}$ such that there exist not necessarily distinct vertices $x_1, x_2, ..., x_{k+1}$ $(x_1, x_2, ...)$ such that either $a_i = (x_i, x_{i+1})$ for all $i = 1, 2, ..., k$ $(i = 1, 2, ...)$ or $a_i = (x_{i+1}, x_i)$ for all $i = 1, 2, ..., k$ $(i = 1, 2, ...)$.

Note that paths in digraphs considered above need not be simple. An increasing path is a sequence of arcs $(a_1, a_2) \in A(\mathcal{G})$ where $x_i \neq x_{i+1}$ iff $i \neq j$. By a cycle in a digraph we understand a sequence of arcs $(a_1, a_2), (a_2, a_3), ..., (a_j, a_k+1), (a_k+1, a_1)$. Let us recall the definition of the classes $\mathcal{E}_{\text{fin}}, \mathcal{V}_{\text{inf}}, \mathcal{V}_{\text{fin}}, \mathcal{E}_{\text{inf}}$ given in introduction. We say that a graph belongs to $\mathcal{V}_{\text{inf}}$ (resp. $\mathcal{E}_{\text{inf}}$) if for every injective mapping $f: V(\mathcal{G}) \rightarrow L$ (resp. $g: E(\mathcal{G}) \rightarrow L$), where $L$ is an arbitrary linearly ordered set, there exists an infinite path $x_1, x_2, ...$ in $\mathcal{G}$ such that either $f(x_1) < f(x_2) < ...$ or $f(x_1) > f(x_2) > ...$ (resp. either $g(\{x_1, x_2\}) < g(\{x_2, x_3\}) < ...$ or $g(\{x_1, x_2\}) > g(\{x_2, x_3\}) > ...$).

We say that $\mathcal{G}$ belongs to $\mathcal{V}_{\text{inf}}$ (resp. $\mathcal{E}_{\text{inf}}$) if for every $f$ (resp. $g$ as above there exist arbitrarily long finite paths $x_1, x_2, ..., x_n$ in $\mathcal{G}$ such that $f(x_1) < f(x_2) < ... < f(x_n)$, (resp. $g(\{x_1, x_2\}) < g(\{x_2, x_3\}) < ... < g(\{x_{k-1}, x_k\})$). Such paths we call $f$-monotone and $g$-monotone respectively.

Remark. One can easily see that the following two statements are equivalent:

1) $\mathcal{G} \in \mathcal{V}_{\text{fin}}$,
2) for every mapping $f: V(\mathcal{G}) \rightarrow L$ where $L$ is an arbitrary linearly ordered set and for every $n$ there exists a path $x_1, x_2, ..., x_n$ in $\mathcal{G}$ such that $f(x_1) \equiv f(x_2) \equiv ... \equiv f(x_n)$.

The analogous statements hold for the classes $\mathcal{E}_{\text{inf}}, \mathcal{V}_{\text{inf}}$, and $\mathcal{E}_{\text{inf}}$ as well.