Remark on the Weierstrass Points on Curves

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Abstract. We give a treatment of the Weierstrass points of curves which is a little different from the treatment by Laksov. We introduce the notion of the i-th weight which makes the treatment easier and gives an algorithm for computing the gap sequence of an effective divisor and the weight at a point.

§1. Introduction

Let $C$ be a smooth curve over an algebraically closed field of arbitrary characteristic and $L$ a line bundle on $C$. Laksov gave a complete treatment of the theory of Weierstrass points of $L$ on $C$ in [1], [2]. In this paper we try to give a different version of his treatment by introducing the $i$-th differential line bundles of $L$. The difference is inessential, but our treatment is simpler in my point of view, for example, so far as the discussion on the Wronskian is concerned. Besides, we define the (total) $i$-th weight at any point and give an algorithm for computing the gap sequence of $L$ and the weight at a point. Though we could give an independent treatment without referring to [1], [2] we prefer to use them since actually our main idea comes from them.

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§2. Weierstrass points

We shall adopt most of the notation in [1], [2] and [3].

Let $C$ be as above and $D$ a positive divisor with $L = O(D)$ of degree $d$ and dimension $r$. Denote by $P^m(D)$ the sheaf of $m$-th principal parts of $D$ and by $a_m : V_c = H^0(C, L) \otimes_k O_c \to P^m(L)$ the $m$-th canonical homomorphism. Let $B^m = \text{Im}a_m$ and $A^m = \text{coker } a_m$ (see [1] §1 for details).
Now we have the diagrams

\[
\begin{array}{ccc}
V_c & \rightarrow & B^m \\
\downarrow b^m & & \downarrow \\
V_c & \rightarrow & B^{m-1} \\
& & 0
\end{array}
\]

and

\[
\begin{array}{ccc}
0 & \rightarrow & G^m \\
\downarrow & & \downarrow \\
0 & \rightarrow & P_c^m(L) \\
\downarrow & & \downarrow \\
0 & \rightarrow & P_c^{m-1}(D) \\
& & 0
\end{array}
\]

where \( b^m \) is induced by \( P^m \rightarrow P^{m-1} \) and \( G^m = \ker b^m \). Note that \( G^0 = B^0 \) and \( B^m \), and hence the \( G^m \) are locally free.

**Proposition 2.1.** There are integers \( 0 = b_0 < b_1 < \ldots < b_r \leq d < \infty \) such that \( \text{rk} G^j = 0 \) for \( j \neq b_i \) and \( \text{rk} G^{b_i} = 1 \) for all \( b_i \).

**Proof.** Since there exists a sequence \( 0 = b_0 < \ldots < b_r \leq d < b_{r+1} = \infty \) such that if \( b_m < j < b_{m+1} \) then \( \text{rk} B^j = m + 1 \) by [1], so we have the conclusion.

We call \( b_0, b_1, \ldots, b_r \) the gap sequence of \( L \) which is the same as that in [1].

**Remark.** If we replace \( V \) by a subspace \( V' \subset V \) with \( \dim V' = 1 + r' \), we shall get a corresponding sequence \( b_0, \ldots, b_r' \) in the same way. We still call it the gap sequence for the linear system \( V' \).

**Definition.** Let \( G^{b_i} = G_i \). We call \( G_i \) the \( i \)-th differential line bundle of \( L \) (or of \( V' \)).

For \( G_i \) we have a natural morphism

\[
h_i : 0 \rightarrow G_i \rightarrow \Omega^2_{\mathbf{P}^1}(D)
\]

induced by \( a_{b_i} \).

**Definition.** Let \( D_i \) be the degeneracy of \( h_i \) defined by the Fitting ideal \( F^0 \) (coker \( h_i \)). For any point \( x \in C \) we call the multiplicity \( w_i(x) \) of \( x \) in \( D_i \) the \( i \)-th weight of \( x \) with respect to \( L \) (or \( V' \)) and \( w(x) = \sum_{i=0}^r w_i(x) \) the weight of \( x \) and \( W = \sum_{x \in C} w(x) \) the total weight.

A point \( x \) is a **Weierstrass point** for \( L \) (resp. a Wronski point for \( V' \)) if for some \( i, w_i(x) \neq 0 \). \( I = \bigcup_{i=0}^r D_i \) as a scheme-theoretic union is called the **Weierstrass locus** of \( L \) (resp. the Wronski locus of \( V' \)), and the invertible sheaf corresponding to \( I \) is called the **Wronskian** for \( I \) (resp. for \( V' \)).

**Proposition 2.2.** The Weierstrass point, the Wronskian and the total weight defined above, all of them are the same as those defined in [1], [2].