On the Representation Theory of Möbius Groups in $R^n$*

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Abstract. We will solve several fundamental problems of Möbius groups $M(R^n)$ which have been matters of interest such as the conjugate classification, the establishment of a standard form without finding the fixed points and a simple discrimination method.

Let $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a Clifford matrix of dimension $n, c \neq 0$. We give a complete conjugate classification and prove the following necessary and sufficient conditions: $g$ is f.p.f. (fixed points free) iff $g \sim \begin{bmatrix} \alpha & \beta \\ c & \alpha^t \end{bmatrix}$, $|\alpha| < 1$ and $|E - AE^1| \neq 0$; $g$ is elliptic iff $g \sim \begin{bmatrix} \alpha & \beta \\ c & \alpha^t \end{bmatrix}$, $|\alpha| < 1$ and $|E - AE^1| = 0$; $g$ is parabolic iff $g \sim \begin{bmatrix} \alpha & 0 \\ c & \alpha^t \end{bmatrix}$, $|\alpha| = 1$; and $g$ is loxodromic iff $g \sim \begin{bmatrix} \alpha & \beta \\ c & \alpha^t \end{bmatrix}$, $|\alpha| > 1$ or rank $(E - AE^1) \neq \text{rank}(E - AE^1, \alpha c^{-1} + c^{-1}d)$, where $\alpha$ is represented by the solutions of certain linear algebraic equations and satisfies

$$|c^{-1}\alpha^t| = ||(E - AE^1)^{-1}(\alpha c^{-1} + c^{-1}d)||.$$

§1. Introduction

1. Let $M(R^n)$ be orientation-preserving Möbius group in $R^n$. N. J. Wielenberg\[1\] classified elements in $M(R^n)$ in terms of their fixed points into parabolic, elliptic and loxodromic in 1977. S. Agard\[2\] also classified elements in $M(R^n)$ into these three classes by Poincaré extension in $R^{n+1}$ in 1983. This classification is complete. We will give a refinement of these classifications.

K. Th. Vahlen established a Clifford matrix representation of the elements of $M(R^n)$ in 1902, but his paper had been forgotten until revived by H. Mass in 1949, except for an

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unfavorable mention in an encyclopedia article by E. Cartan. Since 1984, L. Ahlfors\cite{3,4,5} has expounded Vahlen's theorem again and again, emphasized its importance and gave a sketch accompanied with a very high appraisal. In 1985, he defined hyperbolic, elliptic and parabolic transformations by the conjugate class and gave their identifications; he pointed out that his definition for ellipticity is ambiguous. But he didn't give a complete conjugate classification and a standard form. Recently, F. W. Gehring\cite{6} established a complete classification of the discrete convergence groups, but he didn't give a complete classification of the convergence groups, while $M(R^n)$ itself is just a convergence group.

2. Since 1986, we begin the study of fixed points and classifications of elements in $M(R^n)$. In 1987, Zhou Zhan [*1] found an example without fixed points in $M(R^n)$ and complemented Ahlfors' classifications, but he didn't go further. In 1988, Liu Chun Lin\cite{7} established a necessary and sufficient condition of the elements of $M(R^3)$ without fixed points by quaternions and gave a complete conjugate classification with identifications. When $n \geq 4$, his method can not be used for $M(R^n)$. In 1989, Wang Xian Tao [*2] established a complete classification of f.p.f., loxodromic, parabolic and elliptic elements by their fixed points in $M(R^n)$ and identifications by the Descartes matrix representation formula. We \cite{8} established the relations between Clifford matrices and Descartes matrix representation formulas, gave a detailed proof of the Vahlen's theorem. We \cite{9} studied the invariant balls of elements and the more careful classifications of the loxodromic and parabolic elements in $M(R^n)$, proved that the loxodromic elements in $M(R^{2k+1})$ certainly have an invariant ball, expounded the geometric meaning of Ahlfors' hyperbolic elements, introduced uniformly hyperbolic and parabolic elements and gave their identifications. We found the ranges of $\text{Re}(\alpha + \beta^*)$ for each class of $M(R^n)$.

In this paper, we study the representation theory of $M(R^n)$ by the Clifford matrix. We establish a necessary and sufficient condition for elements without fixed points, a complete conjugate classification, their identifications and conjugate standard form through solving a linear algebraic equation system without finding the fixed points.

3. Throughout this paper, we adopt the same notations as in [5]. Let $M_n$ be a Clifford matrix group of dimension $n$, $A(z)$ (or $A$) be an orthogonal matrix in the formula $czc^{-1} = Az$ with $|A| = 1$, $E^1 = \begin{bmatrix} -1 & 0 \\ 0 & E_{n-1} \end{bmatrix}$, $E_{n-1}$ be an $(n-1) \times (n-1)$ identity matrix, $x_\pm = x \pm x_n e_n$ with $x \in R^n$ and $\infty > x_n > 0$, $H^n = \{x : x = x_0 + x_1 e_1 + \cdots + x_{n-1} e_{n-1} \in R^n, x_n > 0\}$ and $\overline{g}(z)$ be Poincaré extension in $R^{n+1}$ of $g(z)$ in $M(R^n)$. For more advanced materials on Clifford algebra, see [3-5], [10].

Because we concern ourselves mainly with the conjugate invariant properties of $M(R^n)$ and its elements, thus without loss of generality, throughout this paper we put $c \neq 0$ if we do not state otherwise.