On the Representation Theory of Möbius Groups in $\mathbb{R}^n$

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Abstract. We will solve several fundamental problems of Möbius groups $M(\mathbb{R}^n)$ which have been matters of interest such as the conjugate classification, the establishment of a standard form without finding the fixed points and a simple discrimination method.

Let $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a Clifford matrix of dimension $n, c \neq 0$. We give a complete conjugate classification and prove the following necessary and sufficient conditions: $g$ is f.p.f. (fixed points free) iff $g \sim \begin{bmatrix} \alpha & \beta \\ c & \alpha' \end{bmatrix}, |\alpha| < 1$ and $|E - \alpha E^1| 
eq 0; g$ is elliptic iff $g \sim \begin{bmatrix} \alpha & \beta \\ c & \alpha' \end{bmatrix}, |\alpha| < 1$ and $|E - \alpha E^1| = 0; g$ is parabolic iff $g \sim \begin{bmatrix} \alpha & 0 \\ c & \alpha' \end{bmatrix}, |\alpha| = 1; g$ is loxodromic iff $g \sim \begin{bmatrix} \alpha & \beta \\ c & \alpha' \end{bmatrix}, |\alpha| > 1$ or rank $(E - \alpha E^1) \neq $ rank$(E - \alpha E^1, \alpha c^{-1} + c^{-1}d)$, where $\alpha$ is represented by the solutions of certain linear algebraic equations and satisfies

$$|c^{-1} \alpha'| = |(E - \alpha E^1)^{-1} (\alpha c^{-1} + c^{-1}d)|.$$

§1. Introduction

1. Let $M(\mathbb{R}^n)$ be orientation-preserving Möbius group in $\mathbb{R}^n$. N. J. Wielenberg$^{[1]}$ classified elements in $M(\mathbb{R}^n)$ in terms of their fixed points into parabolic, elliptic and loxodromic in 1977. S. Agard$^{[2]}$ also classified elements in $M(\mathbb{R}^n)$ into these three classes by Poincaré extension in $\mathbb{R}^{n+1}$ in 1983. This classification is complete. We will give a refinement of these classifications.

K. Th. Vahlen established a Clifford matrix representation of the elements of $M(\mathbb{R}^n)$ in 1902, but his paper had been forgotten until revived by H. Mass in 1949, except for an

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unfavorable mention in an encyclopedia article by E. Cartan. Since 1984, L. Ahlfors has expounded Vahlen's theorem again and again, emphasized its importance and gave a sketch accompanied with a very high appraisal. In 1985, he defined hyperbolic, elliptic and parabolic transformations by the conjugate class and gave their identifications; he pointed out that his definition for ellipticity is ambiguous. But he didn't give a complete conjugate classification and a standard form. Recently, F. W. Gehring established a complete classification of the discrete convergence groups, but he didn't give a complete classification of the convergence groups, while \( M(R^n) \) itself is just a convergence group.

2. Since 1986, we begin the study of fixed points and classifications of elements in \( M(R^n) \). In 1987, Zhou Zhan found an example without fixed points in \( M(R^n) \) and complemented Ahlfors' classifications, but he didn't go further. In 1988, Liu Chun Lin established a necessary and sufficient condition of the elements of \( M(R^3) \) without fixed points by quaternions and gave a complete conjugate classification with identifications. When \( n \geq 4 \), his method can not be used for \( M(R^n) \). In 1989, Wang Xian Tao established a complete classification of f.p.f., loxodromic, parabolic and elliptic elements by their fixed points in \( M(R^n) \) and identifications by the Descartes matrix representation formula. We established the relations between Clifford matrices and Descartes matrix representation formulas, gave a detailed proof of the Vahlen's theorem. We studied the invariant balls of elements and the more careful classifications of the loxodromic and parabolic elements in \( M(R^n) \), proved that the loxodromic elements in \( M(R^{2k+1}) \) certainly have an invariant ball, expounded the geometric meaning of Ahlfors' hyperbolic elements, introduced uniformly hyperbolic and parabolic elements and gave their identifications. We found the ranges of \( \text{Re}(\alpha + \alpha^*) \) for each class of \( M(R^n) \).

In this paper, we study the representation theory of \( M(R^n) \) by the Clifford matrix. We establish a necessary and sufficient condition for elements without fixed points, a complete conjugate classification, their identifications and conjugate standard form through solving a linear algebraic equation system without finding the fixed points.

3. Throughout this paper, we adopt the same notations as in [5]. Let \( M_n \) be a Clifford matrix group of dimension \( n \), \( A(z) \) (or \( A \)) be an orthogonal matrix in the formula \( c^*x_c^{-1} = Ax \) with \( |A| = 1 \), \( E^1 = \begin{bmatrix} -1 & 0 \\ 0 & E_{n-1}\end{bmatrix} \), \( E_{n-1} \) be an \( (n-1) \times (n-1) \) identity matrix, \( x_\pm = x \pm x_n \epsilon_n \) with \( x \in R^n \) and \( \infty > x_n > 0 \), \( H^n = \{ x : x = x_0 + x_1 \epsilon_1 + \cdots + x_{n-1} \epsilon_{n-1} \in R^n, x_{n-1} > 0 \} \) and \( \mathcal{G}(x) \) be Poincaré extension in \( R^{n+1} \) of \( \mathcal{G}(x) \) in \( M(R^n) \). For more advanced materials on Clifford algebra, see [3-5], [10].

Because we concern ourselves mainly with the conjugate invariant properties of \( M(R^n) \) and its elements, thus without loss of generality, throughout this paper we put \( \epsilon \neq 0 \) if we do not state otherwise.