Randomly $n$-Cyclic Digraphs

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Abstract. A digraph $D$ is randomly $n$-cyclic ($n \geq 3$) if for each vertex $v$ of $D$, every (directed) path with initial vertex $v$ and having length at most $n - 1$ can be extended to a $v - v$ (directed) cycle of length $n$. Several results related to and examples of randomly $n$-cyclic digraphs are presented. Also, all randomly $n$-cyclic digraphs for $n = 3, 4, 5$ are determined.

In 1969 Chartrand, Kronk and Lick [2] defined a digraph $D$ to be randomly hamiltonian if for each vertex $v$ of $D$, every (directed) path with initial vertex $v$ can be extended to a (directed) hamiltonian $v - v$ cycle. (See [1] or [4] for basic graph theory terminology.) The randomly hamiltonian digraphs were characterized in [2].

In this article the concept of randomly hamiltonian digraphs is generalized. For $n \geq 3$, a digraph $D$ is defined to be randomly $n$-cyclic if for each vertex $v$ of $D$, every path with initial vertex $v$ and having length at most $n - 1$ can be extended to a $v - v$ cycle of length $n$. Thus a digraph $D$ of order $p \geq 3$ is randomly hamiltonian if and only if $D$ is randomly $p$-cyclic.

For a fixed integer $n \geq 3$, there are numerous examples of randomly $n$-cyclic digraphs. The complete symmetric digraphs $K^*_{p}$ ($p \geq n$), the (directed) cycle $Cs$ and symmetric cycle $Cs^*$ are all randomly $n$-cyclic. The digraphs $D_n$ ($n = 3, 4, 5$) of Fig. 1 are randomly $n$-cyclic.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{randomly_n-cyclic_digraphs.png}
\caption{Randomly $n$-Cyclic Digraphs}
\end{figure}

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In order to present another class of randomly n-cyclic digraphs, we define a digraph $D$ to be cyclically complete $k$-partite ($k \geq 2$) if the vertex set $V(D)$ of $D$ can be partitioned into subsets $V_1, V_2, \ldots, V_k$ such that $(u, v)$ is an arc of $D$ if and only if $u \in V_i$ and $v \in V_j$, for some $i$ and $j$ with $j - i \equiv 1 \pmod{k}$. If $|V_i| = p_i$ ($1 \leq i \leq k$), then this digraph $D$ is denoted by $D(p_1, p_2, \ldots, p_k)$. Note that the digraph $D_4$ of Figure 1 is the cyclically complete 2-partite digraph $D(2, 2)$, while $D_5$ is the cyclically complete 5-partite digraph $D(1, 1, 1, 1, 1)$.

It is not difficult to see that if $n \geq 3$ and $k \geq 2$ are integers such that $k|n$, then for arbitrary integers $p_1, p_2, \ldots, p_k$ with $p_i \geq n/k$ for all $i$ ($1 \leq i \leq k$), then the digraph $D(p_1, p_2, \ldots, p_k)$ is randomly $n$-cyclic. Consequently, the digraph $D(2, 2, 3)$ of Figure 2 is randomly 6-cyclic.

Although we have already seen randomly n-cyclic digraphs that are not cyclically complete $k$-partite digraphs, there is a relationship between these two types of digraphs, as we shall now see.

**Proposition 1.** If $D$ is a connected, randomly n-cyclic digraph of order $p$ ($3 \leq n \leq p$), then $D$ contains a spanning subdigraph $D(p_1, p_2, \ldots, p_n)$ for some positive integers $p_1, p_2, \ldots, p_n$ (where then $\sum_{i=1}^{n} p_i = p$).

**Proof.** Certainly $D$ contains an $n$-cycle so that $D$ contains a subdigraph $D(1, 1, \ldots, 1)$. If $p = n$, then the proof is complete; so assume that $p > n$. Suppose that $D$ contains a subdigraph $H \cong D(r_1, r_2, \ldots, r_n)$, where $\sum_{i=1}^{n} r_i < p$. Since $D$ is connected, there exists a vertex $v_1$ of $D$ such that $v_1 \notin V(H)$ and $v_1$ is adjacent to or adjacent from some vertex of $H$. Suppose that $V(H)$ is partitioned into subsets $U_1, U_2, \ldots, U_n$ such that $|U_i| = r_i$ ($1 \leq i \leq n$) and $(u, v) \in A(H)$ if and only if $u \in U_i$ and $v \in U_j$, for some $i$ and $j$ with $j - i \equiv 1 \pmod{n}$.

Assume, without loss of generality, that $v_1$ is adjacent to a vertex of $U_2$. Let $u \in U_n$. Then there exists a path $v_1, u_2, u_3, \ldots, u_{n-1}, u$ in $D$, where $u_i \in U_i$ for $2 \leq i \leq n - 1$. Since $D$ is randomly $n$-cyclic, the arc $(u, v_1)$ belongs to $D$, i.e., every vertex of $U_n$ is adjacent to $v_1$. Let $w \in U_2$. Then there exists a path $w, u_3, u_4, \ldots, u_n, v_1$, where $u_i \in U_i$ for $3 \leq i \leq n$. Since $D$ is randomly $n$-cyclic, the arc $(v_1, w)$ belongs to $D$, i.e., every vertex of $U_2$ is adjacent from $v_1$. If we let $H' = \langle V(H) \cap \{v_1\} \rangle$, then $H' \cong D(p_1, p_2, \ldots, p_n)$.