Normal Extensions of Semigroups of Operators to Krein Spaces

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Abstract. In this paper, we prove that every strongly continuous semigroup of bounded operators on a Hilbert space may be extended to a strongly continuous semigroup of normal operators on a larger Krein space. Several equivalent formulations for the case where the extension space is a Pontrjagin space are given.

§1. Introduction

In [7] we proved that every bounded linear operator on a Hilbert space has a normal extension to a Krein space. We now turn our attention to the extension problem of semigroups. Let \( \{A_t : t \geq 0\} \) be a strongly continuous semigroup of bounded subnormal operators on a Hilbert space \( H \). Itô \([4]\) proved that there exist a Hilbert space \( K \) containing \( H \) as a subspace, and a strongly continuous semigroup \( \{N_t : t \geq 0\} \) of bounded normal operators on \( K \) such that \( A_t \subseteq N_t \) for all \( t \geq 0 \) and such that \( K \) is spanned by vectors of the form \( N_t^*x \) where \( x \in H \) and \( t \geq 0 \). Suppose now that \( \{T_t : t \geq 0\} \) is a strongly continuous semigroup of bounded operators on a Hilbert space \( H \) and that \( A \) is its infinitesimal generator. The main aim of this paper is to prove that there exist a Krein space \( K \) containing \( D(A) \) as a positive subspace, and a strongly continuous semigroup \( \{T_t : t \geq 0\} \) of bounded normal operators on \( K \) such that \( T_t |_{D(A)} \subseteq T_t \) for all \( t \geq 0 \) and such that \( K \) is spanned by vectors of the form \( T_t^*x \) where \( x \in D(A) \) and \( t \geq 0 \). Roughly speaking, every strongly continuous semigroup of operators has a minimal normal extension to a Krein space. We also consider the case where \( K \) is a Pontrjagin space with a finite rank of negativity. Let \( \{T_t : t \geq 0\} \) be a strongly continuous semigroup of bounded \( J \)-subnormal operators on a Hilbert space \( H \) and let the orders of \( T_t \), \( t \geq 0 \), be bounded. We shall prove that there exist a Pontrjagin space \( K \) with a finite rank of negativity and containing \( H \) as a uniformly positive subspace, and a strongly continuous semigroup \( \{T_t : t \geq 0\} \) of bounded normal operators on \( K \) such that \( T_t |_{K} \subseteq T_t \) for all \( t \geq 0 \) and such that \( \{T_t : t \geq 0\} \) is minimal. Other characterizations besides this are also obtained.

Throughout this paper, by an operator we mean a bounded linear transformation acting on a space, unless it is otherwise stated. The inner product on a Hilbert space is denoted by \( \langle \cdot , \cdot \rangle \). The indefinite inner product on a Krein space is denoted by \( \langle \cdot , \cdot \rangle \). For a fundamental symmetry \( J \), the corresponding \( J \)-inner product is denoted by \( \langle \cdot , \cdot \rangle_J \) (or simply by \( \langle \cdot , \cdot \rangle \)). Let \( T \) be an operator on a Krein space. Denote by \( T^* \) the adjoint of \( T \) with respect to the indefinite inner product \( \langle \cdot , \cdot \rangle \). Other concepts and notation in the theory of indefinite inner product spaces to be used in this paper agree with those
in [1] or [3]. Recall that an operator $T$ on a Hilbert space is $J$-subnormal of order $n$ if there exist a Pontrjagin space $K$ with rank $n$ of negativity and containing $H$ as a positive subspace, and a normal operator $\hat{T}$ on $K$ such that $T = \hat{T}$ and such that $K$ is spanned by vectors of the form $\hat{T}^k x$ where $x \in H$ and $k = 0, 1, 2, \ldots$ The subnormality and the $J$-subnormality of order 0 are equivalent. The theory of semigroups of operators can be found in the classical book of Hille and Phillips [2] or any book that deals with the subject, e.g. [5].

§2. Normal Extension to a Krein Space

The following lemma, together with its proof, is the same as in the case of Hilbert space. See [2, Theorem 22.4.1].

**Lemma 1.** Let $\{N_t : t \geq 0\}$ be a strongly continuous semigroup of normal operators on a Krein space $K$. Then

$$N_s N_t^* = N_t N_s^*$$

for all $s, t > 0$.

**Proof.** Suppose first that $s$ and $t$ are commensurate. There exist positive integers $k$ and $m$ and $u > 0$ such that $s = ku$ and $t = mu$. The normality of $N_u$ implies that

$$N_s N_t^* = (N_u)^m (N_u^*)^k = (N_u^*)^k (N_u)^m = N_t N_s^*.$$ 

For arbitrary $s, t > 0$, we choose a sequence $\{t_j\}$ of positive numbers such that $t_j$ is commensurate with $s$ and such that $t_j \to t$. The strong continuity of the semigroup implies that

$$N_s N_t^* x = \lim_{j \to \infty} N_{t_j} N_s^* x = \lim_{j \to \infty} N_s N_{t_j}^* x = N_s N_t x$$

for all $x \in K$.

**Theorem 2.** Let $\{T_t : t \geq 0\}$ be a strongly continuous semigroup of operators on a Hilbert space $H$ and let $A$ be its infinitesimal generator. Then there exist a Krein space $K$ containing $D(A)$ as a positive subspace, and a strongly continuous semigroup $\{\widetilde{T}_t : t \geq 0\}$ of normal operators on $K$ such that $T_t | D(A) \subset \widetilde{T}_t$, for all $t \geq 0$ and such that $\{\widetilde{T}_t : t \geq 0\}$ is minimal. More precisely, the following assertions hold:

(a) $\langle x, y \rangle = \langle x, y \rangle$ for all $x, y \in D(A)$;
(b) $T_t x = \tilde{T}_t x$ for all $x \in D(A)$ and $t \geq 0$;
(c) $\tilde{T}_s \tilde{T}_t = \tilde{T}_{s+t}$ for all $s, t \geq 0$;
(d) $K$ is spanned by vectors of the form $\tilde{T}_t x$ where $x \in D(A)$ and $t \geq 0$;
(e) $D(A) \subset D(\tilde{A})$ where $\tilde{A}$ is the infinitesimal generator of $\{\tilde{T}_t : t \geq 0\}$.

**Proof.** Without loss of generality, we may assume that the type of $\{T_t : t \geq 0\}$ is negative. Otherwise, we consider $S_t = e^{-\beta t} T_t$ with $\beta$ being chosen such that this condition is satisfied. Thus there exist constants $M > 0$ and $\omega < 0$ such that

$$\| T_t \| \leq M e^{\omega t} \quad (\forall t \geq 0).$$

Define $\overline{T}$ on $L^2(0, + \infty; H)$ by

$$\overline{T}\xi(t) = \int_0^\infty T_t^* T_s \xi(s) ds \quad (\forall \xi \in L^2(0, + \infty; H)). \quad (1)$$

We have