INTEGRABLE EQUATIONS ON Z-GRADED LIE ALGEBRAS

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We consider evolution systems admitting L–A-pairs in Z-graded Lie algebras. We relate several hierarchies of integrable systems to a single L operator. The different hierarchies correspond to different decompositions of the zero component of a Z-graded algebra into the sum of two subalgebras. This allows us to construct new examples of multi-component integrable systems following the Burgers, mKdV, NLS, and Boussinesq equations.

It was shown in [1] how, given the operator

\[ L = D_x + a\lambda + q(x, t), \]

where \( \lambda \) is the spectral parameter, \( q \) belongs to a Lie algebra \( \mathfrak{g} \), and \( a \) is a constant element of \( \mathfrak{g} \), one can construct an operator \( A = \sum_{i=0}^{n} p_i \lambda^i \) such that the operator relation \( L_t = [A, L] \) is equivalent to the evolution equation of the form

\[ q_t = F(q, q_x, q_{xx}, \ldots ). \]

The method given in [1], which is based on bringing operator \( L \) to a diagonal form, allows one to constructively build, in addition to the \( A \) operator, the higher symmetries and conservation laws for Eq. (2).

The construction of [1] can be easily generalized to the case of \( L \) operators of a more general form:

\[ L = D_x + a\lambda^{n+1} + \sum_{i=-m}^{n} q_i(x, t)\lambda^i. \]

In this paper, instead of proceeding as in [1], where the decomposition of the Lie algebra of the Laurent series \( \mathfrak{g}[[\lambda, \lambda^{-1}]] \) into the sum of a polynomial in \( \lambda \) and a series containing only negative powers of \( \lambda \) was used, we build the operator \( A \) using a more involved construction related to the decomposition of the \( \mathfrak{g} \) algebra into a sum of subspaces, each of which is a subalgebra in \( \mathfrak{g} \). This allows us to construct several different \( A \) operators for a given \( L \) and, thus, to considerably broaden the list of integrable equations (2).

In order to include equations over the Kac–Moody algebras [1] in our analysis, as well as the matrix and vector integrable systems considered in [2–9], we follow [10] by replacing the Laurent polynomials with arbitrary Z-graded Lie algebras.

1. The general scheme. Let us remind the reader that the Lie algebra \( \mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i \) is called Z-graded if \( \mathfrak{g}_i \) are vector subspaces such that \( [\mathfrak{g}_k, \mathfrak{g}_j] \subseteq \mathfrak{g}_{k+j} \). It is clear that \( \mathfrak{g}_0 \) is a subalgebra of \( \mathfrak{g} \).

Let us assume that

\[ \mathfrak{g}_0 = \mathfrak{A}_+ \oplus \mathfrak{A}_- \]

is a direct sum of vector subspaces that are Lie subalgebras. If \( \mathfrak{g}_0 \) is a full matrix algebra, the decompositions into the sum of lower- and upper-triangular matrices and also into the sum of upper-triangular and skew-symmetric ones would be the most standard.

In the examples given in what follows, \( \mathfrak{g}_0 \) is the algebra of all diagonal matrices whose elements are from an arbitrary associative ring \( \mathbb{R} \). The corresponding Eq. (2) is, then, one or several equations for the

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unknown \mathfrak{g}-valued functions. We call such equations non-Abelian. The coordinate form of a non-Abelian equation gives rise, of course, to a conventional multi-component system of evolution equations.

Whenever \mathfrak{g} is taken as the full matrix algebra \( gl(k) \) in our examples, the set \( \mathfrak{g}_0 \) is the set of block-diagonal matrices. In the simplest case of two blocks, the problem of decomposing \( \mathfrak{g}_0 \) into a direct sum of its subalgebras is intimately related to constant solutions of the modified classical Yang–Baxter equation for \( gl(k) \) (see [11, 12]):

\[
R([R(X),Y] - [R(Y),X]) = [X,Y] + [R(X),R(Y)], \quad X,Y \in gl(k).
\]

Namely, for any operator \( R \) satisfying (4), the decomposition we seek is given by

\[
\mathfrak{A}_{-} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right\} \quad \text{and} \quad \mathfrak{A}_{+} = \left\{ \begin{pmatrix} a + R(a) & 0 \\ 0 & -a + R(a) \end{pmatrix} \right\}.
\]

Let the element \( \alpha \in \mathfrak{g}_1 \) satisfy the condition

\[
\text{Ker}(\text{ad} \alpha) \oplus \text{Im}(\text{ad} \alpha) = \mathfrak{g}.
\]

Proposition 1. There exist unique elements \( u = \sum_{i=-\infty}^{-1} u_i, u_i \in \text{Im}(\text{ad} \alpha) \cap \mathfrak{g}_i, \) and \( h = \sum_{i=-\infty}^{0} h_i, h_i \in \text{Ker}(\text{ad} \alpha) \cap \mathfrak{g}_i, \) such that

\[
\exp(\text{ad} u)L = D_x + \alpha + h.
\]

Remark 1. The elements \( u_i \) and \( h_i \) are differential polynomials in \( q \). More precisely, the coefficients of the decomposition of \( u_i \) and \( h_i \) with respect to a basis in \( \mathfrak{g} \) are polynomials in \( q \) and its \( x \)-derivatives.

Proposition 2. Let \( \beta \) be a constant element from the center of subalgebra \( \text{Ker}(\text{ad} \alpha) \).

\[
A_{\beta} = (\exp(-\text{ad} u)(\beta))_{+},
\]

where "+" denotes the projection onto \( \mathfrak{g}_+ \) parallel to \( \mathfrak{g}_- \). Then, \( [A_{\beta}, L] \in \sum_{-m}^{n} \mathfrak{g}_i \).

Remark 2. The claim of Proposition 2 is that relation \( L_t = [A_{\beta}, L] \) is equivalent to some evolution system for the unknown \( q(x,t) \). It is not difficult to see that in the simplest case, where \( \mathfrak{A}_{-} = \mathfrak{g}_0 \) and \( \mathfrak{A}_{+} = \{0\} \), the right-hand side of the equation for \( q \) belongs to \( \text{Im}(\text{ad} \alpha) \) and it may be assumed that \( q(x,t) \in \text{Im}(\text{ad} \alpha) \). In addition, the reduction to \( q \in \mathfrak{A}_{+} \) is always possible.

The following statements, which can be proved in much the same way as the corresponding theorems from [1], demonstrate that the evolution systems thus constructed possess higher symmetries and conservation laws.