C² STABILITY OF CURVES WITH NON-DEGENERATE SOLUTION TO PLATEAU'S PROBLEM

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Let \( \Gamma^k, k \geq 1 \), be the set of \( C^k \) Jordan curves in \( \mathbb{R}^n \) with its natural topology and let \( \eta: \Gamma^1 \rightarrow \mathbb{N}^*, \mathbb{N}^* = \{1, 2, \ldots, \infty\} \) be the function that assigns to each \( \gamma \in \Gamma^1 \) the number of solutions to Plateau's problem for \( \gamma \), that is, the number of minimal disks bounding \( \gamma \). It is still an unanswered question whether \( \eta \) can reach the value \( \infty \). Several people were able to find open and dense subsets of \( \Gamma^k \) for which \( \eta \) is finite. A result in this direction can be found in [3] where it is proved that there exists an open and dense subset of \( \Gamma^\infty = \bigcap_{k=1}^{\infty} \Gamma^k \), where \( \eta \) is finite. Generally, the approach used for this problem assumes \( k \) large. Consider, for example, the subset \( \Gamma_k \subset \Gamma^k \) of curves whose solutions to Plateau's problem are immersions. In this case A. Tromba [13] was able to show that there exists a subset \( \Gamma_k^1 \) of \( \Gamma_k \) open and dense in \( \Gamma_k \) for \( k \geq 7 \) where \( \eta \) is finite.

The aim of this paper is to present an elementary approach that also works for \( k \geq 2 \) and arbitrary \( n \). In fact, we prove in §4 that there exists an open subset \( \Gamma_k^2 \) of \( \Gamma_2 \) where \( \eta \) is finite and continuous (see theorem (4.1) and corollaries (4.5-6)).

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A similar approach is used in §5 to prove that \( \Gamma_k \) (the intersection of \( \Gamma_k^i \) with \( \Gamma_k \)) is open and dense in \( \Gamma_k \) for \( k \geq 2 \).

This approach also produces regularity results in a natural way. We prove in §3 (see theorem (3.1)) that for \( \gamma \in \Gamma_k \), \( k \geq 2 \), the solutions to Plateau's problem for \( \gamma \) lie in the Sobolev space \( H^{k+1/2}(\mathbb{D}, \mathbb{R}^n) \) where \( \mathbb{D} \) is the unit disk of the plane with center at the origin.

The techniques here arose from a characterization of solutions to Plateau's problem as zeroes of the function \( \psi \) defined in (1.7). This function \( \psi \) is the main tool in [4].

§1. Preliminaries

In this work we use \( u \) and \( v \) for the coordinates of the plane and we denote a complex number by \( z = u + iv \), or, in polar coordinates, as \( z = re^{i\theta} \), where \( i^2 = -1 \). The partial derivative with respect to \( u \), for example, is \( \partial/\partial u \). We also use the following operators:

\[
\mathfrak{A} = \frac{1}{2} \left( \frac{\partial}{\partial u} - i \frac{\partial}{\partial \theta} \right),
\]

\[
\mathfrak{B} = \frac{1}{2} \left( \frac{\partial}{\partial u} + i \frac{\partial}{\partial \theta} \right),
\]

\[
2z\mathfrak{A} = r \frac{\partial}{\partial r} - i \frac{\partial}{\partial \theta}.
\]

In general, we denote by \( df \) the derivative of the map \( f \), but if the domain of \( f \) is an interval then we use \( f' \). We use also \( f_{\theta} \) instead of \( [f(e^{i\theta})]' \) where \( e^{i\theta} = \cos \theta + i \sin \theta \), \( \theta \in \mathbb{R} \).

Let \( M \) be a \( C^\infty \) manifold of dimension \( m \). We will consider the two following families of function spaces: the space \( C^k(M, \mathbb{R}^n) \) of \( C^k \) maps \( f: M \rightarrow \mathbb{R}^n \) with finite \( C^k \) norm, where \( k \) is a non-negative real number, and the Sobolev space \( H^k(M, \mathbb{R}^n) \), \( k \in \mathbb{R} \), defined in [10] as \( L^2_k(M \times \mathbb{R}^n) \). In our case, the manifold \( M \) will be very simple, namely the disk \( D = \{ z/|z| < 1 \} \) or its boundary \( S \). In the later case, the \( H^k \) norm of \( f \in H^k(S, \mathbb{R}^n) \) is