ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF A NONLINEAR WAVE EQUATION

ORLANDO LOPES

Introduction and Statement of the Main Result.

In this paper we deal with the damped nonlinear wave equation:

(1) \[ u_{tt} - \Delta u + cu_t + f(u) = h(t,x), \quad c > 0. \]

We assume \( u(t,x) \) and \( h(t,x) \) are defined for all \( x = (x_1, x_2, x_3) \) in \( \mathbb{R}^3 \) and are \( 2\pi \)-periodic in each \( x_i \); in other words, we take \( 2\pi \)-periodicity in the spatial variables as boundary condition. For each non-negative integer \( k \) and \( 1 \leq p \leq \infty \), \( H_{k,p}(\mathbb{R}^3) \) (\( H_k(\mathbb{R}^3) \) for \( p = 2 \)) denotes the usual Sobolev spaces with the usual norm; \( H_{k,p}^{2\pi}(\mathbb{R}^3) \) (\( H_k^{2\pi}(\mathbb{R}^3) \) for \( p = 2 \)) denotes the Sobolev spaces of functions which are \( 2\pi \)-periodic in each variable (of course, the integrals defining the norm are taken over the fundamental cube \([0,2\pi] \times [0,2\pi] \times [0,2\pi]\)).

Equation (1) can be viewed as a system

(2) \[ \begin{aligned} u_t &= v, \\ v_t &= \Delta u - cu_t + f(u) - h(t,x) \end{aligned} \]

or, more compactly,

(3) \[ \frac{d\omega}{dt} = A\omega + G(t,\omega), \quad \text{where} \]

\[ \omega = (u,v), \quad A\omega = (v,\Delta u), \quad G(t,\omega) = \begin{bmatrix} 0 \\ -cu - f(u) + h(t,\cdot) \end{bmatrix}. \]

Received on 01/08/88.
It is very well known that $A$ generates a strongly continuous semigroup (actually a group) of linear operators in the space $X_k = H^{2\pi}_{k+1} \times H^{2\pi}_k$, for each integer $k \geq 0$. As phase space for equation (3) we take the space $X_1 = H^2_2 \times H^2_1$. Since $L^2_\omega \subseteq H^2_2$ continuously, it is easy to see that $\omega \in X \rightarrow G(t, \omega)6X_1$ is Lipschitzian on bounded sets provided $f$ is $C^2$. So, if this is the case and $h: \mathbb{R}_+ \rightarrow H_1$ is continuous, it follows that local existence and uniqueness of mild solutions of (3) is guaranted in the space $X_1$; moreover, global existence in time can also be guaranted provided we get an a priori estimate for the norm of the mild solutions in the space $X_1$. For $t \geq t_0 \geq 0$ and $\omega_0$ belonging to $X_1$, we denote by $\omega(t, t_0; \omega_0)$ the solution satisfying $\omega(t_0, t_0; \omega_0) = \omega_0$.

**Definition.** Equation (3) is uniform ultimately bounded in the space $X_1$ if there are functions $a(R)$ and $T(R)$ and a constant $H$ such that $|\omega_0|_{X_1} \leq R$ implies $|\omega(t, t_0; \omega_0)|_{X_1} \leq a(R)$ for $t \geq t_0$ and $|\omega(t, t_0; \omega_0)|_{X_1} \leq R_0$ for $t \geq t_0 + T(R)$.

Our main result is the following:

**Theorem A.** Equation (3) is uniform ultimately bounded in the space $X_1$ provided the following conditions are satisfied.

(i) $f(u)$ is a $C^2$-function;

(ii) there are constants $k_1 > 0$ and $k_2$ such that $uf(u) \geq k_1 u^2 + k_2$;

(iii) there are positive constants $k_3$ and $\beta$, $0 \leq \beta < 4$ such that $|f'(u)| \leq k_3 (1 + |u|^\beta)$;

(iv) the map $t \in \mathbb{R}_+ \rightarrow h(t)$ $6 H^2_1$ is continuous and bounded.

Moreover, if the map $t \rightarrow h(t)$ is periodic of period $p > 0$, then the Poincaré map $\omega_0 \rightarrow \omega(p, 0; \omega_0)$ is the sum of a linear map