Active sums of profinite groups

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Dedicated to Professor Philip Hall,
on his eightieth birthday.

Introduction.

In this paper, we extend the construction of active sums of active quivers of groups to the similar construction in categories of pro-$\mathcal{C}$-groups. Originally, the construction was considered by Tomás [6] in the particular case of active normal families of groups. We extended it in [4] for active quivers for groups. Diaz-Barriga & López indicated in [1] how to make the construction for active partially ordered families of profinite groups.

We apply this construction to Galois groups of number fields, showing that the natural homomorphism from the profinite group active sum of the decomposition groups, to the Galois group, is always surjective.

1. Active Sums of Pro-$\mathcal{C}$-Groups.

Let $\mathcal{C}$ be a class of finite groups such that:

1) if $G \in \mathcal{C}$ and $H \leq G$ then $H \in \mathcal{C}$
2) if $G \in \mathcal{C}$ and $H \leq G$ then $G/H \in \mathcal{C}$
3) if $1 \to H \to G \to K \to 1$ is an exact sequence of groups and $H, K \in \mathcal{C}$ then $G \in \mathcal{C}$.

For example, $\mathcal{C}$ may be any one of the following classes: all finite groups; all finite solvable groups; all finite nilpotent groups, all finite abelian groups; all finite cyclic groups; for each prime $p$, all finite $p$-groups.

A pro-$\mathcal{C}$-group is, by definition, any group $G$ which is the inverse (= projective) limit of an inverse system of groups in the class $\mathcal{C}$. Equivalently, a pro-$\mathcal{C}$-group is a profinite group $G$, such that for every open normal subgroup $U$ the group $G/U$ is in the class $\mathcal{C}$.

We consider the category of pro-$\mathcal{C}$-groups, whose objects are the pro-$\mathcal{C}$-groups and whose morphisms are the continuous group-homomorphisms.
We shall extend the constructions in the paper *Active Sums of Groups* [4] to the category of pro-\(\mathcal{C}\)-groups.

Let \(\mathcal{G}\) be a category of groups, for example, the category of all groups, the category of all pro-\(\mathcal{C}\)-groups (where \(\mathcal{C}\) is a class of groups, as indicated). Following [4], we recall the concept of an active quiver of groups of \(\mathcal{G}\).

Let \(\mathcal{J}\) be a directed graph. It consists of a non-empty set \(I\) of vertices and for every \(i,j \in I\) a set \(A(i,j)\) (which may be empty) of arrows. If \(\alpha \in A(i,j)\) then \(i = o(\alpha)\) is the origin of \(\alpha\) and \(j = t(\alpha)\) is the terminal of \(\alpha\). We denote by \(A\) the set of all arrows: \(A = \bigsqcup_{i,j \in I} A(i,j)\). Sometimes we write \(\mathcal{J} = (I, A, o, t)\).

Any non-empty set \(I\) may be viewed as a directed graph, with \(A = \emptyset\). Any partially ordered set \((I, \leq)\) gives rise to a directed graph, with set of vertices \(I\) and

\[
A(i,j) = \begin{cases} \text{set with only one element } \alpha_{ij}, \text{ when } i < j \\ \emptyset, \text{ otherwise} \end{cases}
\]

and \(o(\alpha_{ij}) = i, t(\alpha_{ij}) = j\), when \(i < j\).

A morphism from the directed graph \(\mathcal{J}\) to the directed graph \(\mathcal{J}'\) is a map \(\sigma : I \sqcup A \rightarrow I' \sqcup A'\) such that \(\sigma(I) \subseteq A', \sigma \circ o = o' \circ \sigma, \sigma \circ t = t' \circ \sigma\). A morphism from \(\mathcal{J}\) to \(\mathcal{J}'\) which is bijective is called an automorphism of \(\mathcal{J}\). The set \(\text{Aut}(\mathcal{J})\) of automorphisms of \(\mathcal{J}\) is a group under composition.

Let \(\mathcal{J}\) be a directed graph. A quiver of groups in the category \(\mathcal{G}\), over \(\mathcal{J}\), is a family \(\mathcal{G} = (G_i)_{i \in I}\) of groups (indexed by the set \(I\) of vertices) and for every \(i, j \in I\), a family \((c_{\alpha})_{\alpha \in A(i,j)}\), where \(c_{\alpha} \in \text{Hom}_{\mathcal{G}}(G_i, G_j)\). We assume also (without loss of generality) that if \(\alpha, \beta\) are distinct arrows then \(c_{\alpha} \neq c_{\beta}\). Thus in the case when \(\mathcal{G}\) is the category of pro-\(\mathcal{C}\)-groups, then each \(c_{\alpha}\) is a continuous group-homomorphism.

The spread of the quiver \(\mathcal{G}\) is the set \(\bigsqcup_{i \in I} G_i\), which we denote by \(\bigsqcup \mathcal{G}\).

On \(\bigsqcup \mathcal{G}\) we consider the following partial operation: if \(f, g \in \bigsqcup \mathcal{G}\) and there exists \(i \in I\) such that \(f, g \in G_i\), then \(fg\) is defined and it is equal to the product of the elements \(f, g\) in the group \(G_i\); otherwise, \(fg\) is not defined. On \(\bigsqcup \mathcal{G}\) we consider also the sum of the topologies on the group \(G_i (i \in I)\), so the above operation is continuous, whenever it is defined.

Let \(\mathcal{G}\) be a quiver of groups over \(\mathcal{J}\), let \(\mathcal{G}'\) be a quiver of groups over \(\mathcal{J}'\) (where the groups belong to the category \(\mathcal{G}\)). A morphism from \(\mathcal{G}\) to \(\mathcal{G}'\) is a map \(\sigma : \bigsqcup \mathcal{G} \rightarrow \bigsqcup \mathcal{G}'\) such that there is a morphism \(\sigma : \mathcal{J} \rightarrow \mathcal{J}'\) and the following conditions are satisfied:

1.) for every \(i \in I\), the restriction \(\sigma_i\) of \(\sigma\) to \(G_i\) belongs to \(\text{Hom}_{\mathcal{G}}(G_i, G'_{\sigma(i)})\).
2.) for every \(i, j \in I\) and every \(\alpha \in A(i,j)\), we have \(\sigma_j \circ c_{\alpha} = c'_{\sigma(\alpha)} \circ \sigma_i\).

We say that \(\sigma\) lies over \(\sigma_i\).