The pressure of the geodesic flow on a negatively curved manifold

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Manning [5] has identified the rate of exponential growth of the volume of a ball of radius \( r \) on the universal cover of a compact manifold \( M \) of negative curvature: it is the entropy of the geodesic flow on \( M \). See also Sullivan [7], Chen [3], Chen and Manning [4]. Here we indicate an extension of Manning's result, where the entropy is replaced by the topological pressure \( P(A) \) associated with a function \( A \) on the tangent bundle. It turns out that the Riemann volume used by Manning plays no special role and may be replaced by many other measures.

Let \( M \) be a compact Riemann manifold with strictly negative sectional curvatures everywhere. We denote by \( \tilde{M} \) the universal cover of \( M \) (with the induced metric), by \( p:\tilde{M}\to M \) the canonical projection, and by \( N \) a fundamental domain of finite diameter \( a \). We call \( B(x,r) \) the ball with center \( x \) and radius \( r \) in \( \tilde{M} \). Let \( \mu \) be a positive Radon measure on \( \tilde{M} \), such that there are \( \alpha, \beta, b > 0 \) with

\[
\alpha \leq \mu(B(x,b)) \leq \beta
\]

for all \( x \in \tilde{M} \).

We denote by \( T^{(1)}M \) the unit tangent bundle and let

\[
A:T^{(1)}M\to \mathbb{R}
\]

be a continuous function. For any pair \( x,y \in \tilde{M} \), let \( \sigma(t) \) be the point of abscissa \( t \in [0,d(x,y)] \) on the unique geodesic segment \( xy \) from \( x \) to \( y \).

We define

\[
A_{xy} = \int_{0}^{d(x,y)} A(T_{t}(p\sigma(t)))dt
\]

and, for \( 0 < r_{1} < r_{2} \),

\[
Z(x,r_{1},r_{2}) = \int_{B(x,r_{2})\setminus B(x,r_{1})} \mu(dy) \exp A_{x,y}.
\]

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Theorem. Let $c \geq 2(a+b)$, then

\[ \lim_{r \to \infty} \frac{1}{r} \log Z(x,r,r-c) = P(A) \]

uniformly with respect to $x$, where $P(A)$ is the pressure of $A$ with respect to the geodesic flow $(f^t)$ on $T^{(1)}M$.

Our proof will closely follow that of Manning for the case $A = 0$ (see [5]).

We shall use the formulae (cf. [2])

\[ P(A) = \lim_{\delta \to 0} P^\pm(A,\delta) \]
\[ P^\pm(A,\delta) = \lim_{r \to \infty} \sup \frac{1}{r} Z_r^\pm(A,\delta) \]

\[ Z_r^+(A,\delta) = \sup \left\{ \sum_{\xi \in \mathcal{S}} \exp \int_0^r A(f^t\xi) dt : S \text{ is } (r,\delta) \text{ separated} \right\} \]
\[ Z_r^-(A,\delta) = \inf \left\{ \sum_{\xi \in \mathcal{S}} \exp \int_0^r A(f^t\xi) dt : S \text{ is } (r,\delta) \text{ spanning} \right\} \]

These formulae are easily related to those for the time 1 map $f^1$ and the function $A^1 = \int_0^1 dt \, A \cdot f^t$.

Lemma. Given $\delta, \Delta > 0$ there is $R$ such that if $\sigma, \tau : [0,r] \to \bar{M}$ are two geodesics with $\sigma(0) = \tau(0)$, then $d(\sigma(r), \tau(r)) \leq \Delta$ and $r \geq R$ imply

\[ d(T_t \sigma(t), T_t \tau(t)) \leq \delta \]

in $T^{(1)}\bar{M}$ for $t \in [0,r-R]$.

This is a form of Lemmas 1 and 2 of Manning corresponding to strictly negative curvature: geodesics diverge exponentially.

We shall use the fact, given $\epsilon > 0$, for $d(y,z) \leq$ constant, and sufficiently large $d(x,y)$,

\[ |A_{xy} - A_{xz}| < \frac{1}{2} \epsilon \, d(x,y) \]