ASYMPTOTIC INFERENCE ON A GENERAL MEASURE OF MONOTONE DEPENDENCE

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Summary

In this paper a class of measures of monotone dependence (concordance/discordance) for arbitrary (not necessarily continuous) bivariate distributions is considered. It is shown that the corresponding sampling index of concordance/discordance (which is the most natural estimator of the population index) converges in law to a normal distribution. A Berry-Esseen bound for its rate of convergence is given. Finally, a consistent estimator of the asymptotic variance of the sampling concordance/discordance index is proposed. This last result is essential for constructing confidence intervals and testing hypotheses on the population measure of monotone dependence.

1. Introduction and preliminaries

In recent years general classes of measures of monotone dependence between couples of random variables (r.v.’s) have been proposed (Consonni, 1983; Scarsini, 1984; Cifarelli and Regazzini, 1991; Conti, 1993a). All those measures work out when the two r.v.’s considered are continuous. In this case, general results on their sampling distributions have been obtained (Cifarelli and Regazzini, 1992; Conti, 1993a; Cifarelli, Conti and Regazzini, 1993). In some of those papers (Cifarelli and Regazzini, 1991; Scarsini, 1984; Conti, 1993b), measures of monotone dependence for arbitrary random variables have been analyzed. In the case of discrete finite r.v.’s, those measures can be viewed as ordinal measures for contingency tables, so that their importance for applications is relevant (see Salvemini, 1939, and Coppi, 1969 for some extension of Gini’s and Spearman’s; reviews on ordinal measures for contingency tables are in: Kruskal, 1958; Agresti, 1984). Unfortunately, the descriptive theory of monotone dependence for non-continuous r.v.’s has never received much attention, and general inferential results, comparable to those existing for continuous r.v.’s, have not been worked out so far.

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To our knowledge, measures of concordance which work out in both discrete and continuous cases have been considered by Scarsini (1984), and by Cifarelli and Regazzini (1991). Those measures, which includes Gini's cograduation index and Spearman's rank correlation coefficient as particular cases, have a simple and explicit representation in the continuous case and in the case of paired observations without ties; however, there is no simple representation in the case of non-continuous r.v.'s (e.g., this is the case which originates contingency tables), and the study of their inferential properties is very difficult.

Another general class of measures of monotone dependence, which is suitable for arbitrary r.v.'s, has been recently proposed by Conti (1993b). Those indices include, as particular cases, Gini's index as well as Spearman's coefficient. In the sequel of the present section a short description of such measures is given; for more precise and complete statements, cf. the above mentioned paper.

Let \((X, Y)\) be a bivariate r.v., with joint probability distribution function (p.d.f.) \(H(x, y) = \Pr(X \leq x, Y \leq y)\) and marginal p.d.f.'s \(F(x) = \Pr(X \leq x)\) and \(G(y) = \Pr(Y \leq y)\). Moreover, let \(\tilde{F}(x)\) and \(\tilde{G}(y)\) be equal to \(F(x^-) = \Pr(X < x)\) and \(G(y^-) = \Pr(Y < y)\), respectively. A general measure of monotone dependence between \(X\) and \(Y\) is the following:

\[
\mathcal{M}_f(H) = \mathcal{C}_f(H) - \mathcal{D}_f(H)
\]

where:

\[
\mathcal{C}_f(H) = \mathbb{E}_H[f(F(X), 1 - \tilde{G}(Y)) + f(1 - \tilde{F}(X), G(Y))]
\]

\[
= \int_{\mathbb{R}^2} [f(F(x), 1 - \tilde{G}(y)) + f(1 - \tilde{F}(x), G(y))] \, dH(x, y)
\]

and

\[
\mathcal{D}_f(H) = \mathbb{E}_H[f(F(X), G(Y)) + f(1 - \tilde{F}(x), 1 - \tilde{G}(Y))]
\]

\[
= \int_{\mathbb{R}^2} [f(F(x), G(y)) + f(1 - \tilde{F}(x), 1 - \tilde{G}(y))] \, dH(x, y)
\]

and \(f: [0, 1]^2 \to \mathbb{R}_+\) is a function satisfying appropriate conditions (Conti, 1993b). It has been shown (Conti, 1993b) that the measure \(\mathcal{M}_f(H)\) has all the basic properties characterizing a measure of monotone dependence. The intuitive rationale of the concordance/discordance index \(\mathcal{M}_f(H)\) is simple.