THE GENERALIZED GAUSS MAP OF MINIMAL SURFACES IN $H^3$ AND $H^4$

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1. Introduction

The object of this paper is to establish conditions for a $C^\infty$ map of a Riemann surface $M$ into $Q_{n-2}$, the hyperquadric $z_1^2 + \ldots + z_n^2 = 0$ of $\mathbb{R}^{n-1}$, to be the generalized Gauss map of a minimal conformal immersion of $M$ into $H^3$ and $H^4$, the hyperbolic space of dimensions three and four respectively. Using the upper half-hyperplane as model for the hyperbolic space and exploiting the conformality between the metrics induced on $M$, by the euclidean metric and the hyperbolic metric through the immersion, we can adapt the theory developed by Hoffman and Osserman [H-O,2] to obtain the conditions.

2. Basic facts

Let $\langle , \rangle$ be the usual euclidean metric on $\mathbb{R}^n$ and let $\mathbb{H}^n$ and $\mathbb{H}^n_+$, $n=3,4$, the set $\{(x,t) \mid x \in \mathbb{R}^{n-1}, t > 0\}$ endowed with the metrics $(x,t) = \frac{1}{t^2} \langle , \rangle$ and $\langle , \rangle$ respectively. Given $(x,t) \in \mathbb{H}^n$, let $\hat{x} = (x^1, \ldots, x^{n-1}, 0) \in \mathbb{R}^{n-1}$. Thus $a, b \in \mathbb{H}^n$ implies that $L_{a,b}((x,t)) = \frac{b^n}{a^n} \hat{a}$ is an isometry of $\mathbb{H}^n$ such that $L_{a,b}(a) = b$, $(L_{a,b})^* (v) = \frac{b^n}{a^n} v$ for all $v$ in the tangent space $T_a(\mathbb{H}^n)$. Let $M$ be a Riemann surface and $\tilde{\theta}(p) = (x(p), t(p))$ be a conformal immersion of $M$ into $\mathbb{H}^n$. If

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\( q = (0, \ldots, 0, 1) \in \mathbb{R}^n \), \((L_{\bar{\theta}}(p), q)_*\) sends \( T_{\bar{\theta}}(p)(\mathbb{R}^n) \) isometrically onto \( T_q(\mathbb{R}^n) \), which is \( \mathbb{R}^n \) endowed with its usual inner product.

Let \( G_2(\mathbb{R}^n) \) be the grassmannian of the oriented 2-vector subspaces of \( \mathbb{R}^n \). The map \( \widetilde{G}: M \to G_2(\mathbb{R}^n) \) defined by \( \widetilde{G}(p) = (L_{\bar{\theta}}(p), q)_* \) is \( \widetilde{G}(T_p(M)) \) is the generalized Gauss map of \( \bar{\theta} \). It is well known that \( G_2(\mathbb{R}^n) \) can be identified with the hyperquadric \( Q_{n-2} = \{ [z] \in \mathbb{P}^{n-1} : \sum_{k=1}^{n} z_k^2 = 0 \} \) of the \((n-1)\)-dimensional complex projective space. Such identification will be assumed throughout this paper.

Now let \( z = u + iv \) be local isothermal parameters for \( M \) and let \( \theta \) be a conformal immersion of \( M \) into \( \mathbb{R}^n \) given by \( \theta(p) = \bar{\theta}(p) \) \( \forall \ p \in M \). Then

\[
(2.1) \quad \widetilde{G}(z) = \frac{\partial \theta}{\partial u} - i \frac{\partial \theta}{\partial v} = \frac{\partial \theta}{\partial z},
\]

where \( \frac{\partial \theta}{\partial z} = 1/2 \left( \frac{\partial \theta}{\partial u} - i \frac{\partial \theta}{\partial v}, \ldots, \frac{\partial \theta}{\partial u} - i \frac{\partial \theta}{\partial v} \right) \in \mathbb{C}^n \).

If \( \phi(z) = (\phi_1(z), \ldots, \phi_n(z)) \in \mathbb{C}^n \) is a homogeneous local expression of \( \widetilde{G}(z) \), there is \( \psi: M \to \mathbb{C} - \{0\} \) such that

\[
(2.2) \quad \frac{\partial \psi}{\partial z} = \psi \phi.
\]

Let \( ds^2 \) be the riemannian metric induced on \( M \) by \( \theta \),
\[
\frac{\partial}{\partial z} = 1/2 \left( \frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right), \quad \Delta \text{ the Laplace-Beltrami operator of } M \text{ with respect to } ds^2, \lambda^2 = \left| \frac{\partial \theta}{\partial u} \right|^2 = \left| \frac{\partial \theta}{\partial v} \right|^2, \quad H \text{ the mean curvature vector of } \theta. \text{ It is well known that}
\]

\[
(2.3) \quad \Delta \theta = \frac{4}{\lambda^2} \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial z^2} \theta = 2H.
\]

Indicating by \( \langle , \rangle \) the usual hermitian inner product on \( \mathbb{C}^n \), let

\[
(2.4) \quad V = \phi - n \phi,
\]