A theorem of the Phragmén–Lindelöf type for second-order elliptic operators

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1. Introduction and notations

Let $\mathbb{R}^n$ be the real $n$-dimensional Euclidean space with coordinates $x=(x_1, x_2, \ldots, x_n)$, $|x| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$. $C$ denotes the set of all complex-valued infinitely differentiable functions on $\mathbb{R}^n$ with compact supports and $L^2(\Omega)$ is the Hilbert space of all complex-valued square integrable functions on the set $\Omega$.

In $\mathbb{R}^1$ let $D$ be the domain $\{x | x \geq a\}$ where $a$ is arbitrary and let $L$ be the differential operator

$$-\left(\frac{d}{dx}\right)^2 + \lambda, \lambda > 0.$$ 

The solutions of $Lu = 0$ are

$$u(x) = C_1 e^{\sqrt{\lambda}x} + C_2 e^{-\sqrt{\lambda}x},$$

where $C_1$ and $C_2$ are arbitrary constants. From this we conclude that if a solution is bounded in the domain $D$ or if it belongs to $L^2(D)$ then it decreases like $e^{-\sqrt{\lambda}x}$ when $x$ tends to infinity and the same holds for its derivative. In particular, $ue^{\mu x}$ and $(du/dx)e^{\mu x}$ belong to $L^2(D)$ if $\mu < \sqrt{\lambda}$.

In this paper we shall extend this result to second-order elliptic differential operators in $\mathbb{R}^n$,

$$L = -\sum_{i, k=1}^n a_{ik}(x) D_{ik} + \sum_{k=1}^n b_k(x) D_k + a(x),$$

where $D_{ik} = \partial_i \partial_k + \partial_k \partial_i$, $D_k = \partial_k$ and $a_{ik}(x) = a_{ki}(x)$ (for simplicity we confine ourselves to the real domain).

Giving the result of the general case at the end of the paper we start with the operator $L = -\Delta + a(x)$, where $\Delta$ is the Laplace operator in $\mathbb{R}^n$ and where $a$ is positive and continuous or, more generally, locally bounded and Borel measurable. Then we can prove that if $u$ is a solution of $Lu = 0$ outside some compact set $K$ and if $u$ belongs to $L^2(R^n - K)$ then, in the same sense as above, $u$ and its first derivatives decrease exponentially like $e^{-\varphi(x)}$ when $|x|$ tends to infinity. $\varphi(x)$ is the geodetic distance from the origin to the point $x$ in the metric $ds^2 = a(x)(dx_1^2 + dx_2^2 + \ldots + dx_n^2)$. 

281
L. Lithner, *A theorem of the Phragmén-Lindelöf type*

2. The special case

Let $D$ be the domain $\{x | |x| \geq R\}$, where $R$ is a positive number and let $B$ be the boundary of $D$. $L$ is the operator $-\Delta + a$ where the function $a$ is strictly positive in $R^n$. Let $q(x)$ be the geodetic distance from the origin to the point $x$ in the metric $ds^2 = a(x)(dx_1^2 + dx_2^2 + \ldots + dx_n^2)$ that is, $q(x)$ is the greatest lower bound of

$$\int_{\Gamma} V a(y) V(y_1^2 + dy_2^2 + \ldots + dy_n^2), \quad y = (y_1, y_2, \ldots, y_n),$$

where $\Gamma$ is a piecewise continuously differentiable curve starting at the origin and ending at $x$. Putting further conditions on $q$ will be continuously differentiable.

**Lemma 1.** $|\nabla q(x)| \leq V a(x)$.

**Proof.** It is evident from the definition of $q$ that

$$|q(x + \Delta x) - q(x)| \leq \int_{\Gamma_0} V a(y) \sqrt{\sum dy_i^2},$$

where $\Gamma_0$ is the straight line segment joining $x$ and $x + \Delta x$. This gives the inequality.

**Lemma 2.** (Carleman [1].) If $u$ belongs to $L^2(D)$ and is a solution of $Lu = 0$ then $Vu$ and $|\nabla u|$ belong to $L^2(D)$.

**Proof.** Let $\psi$ be a positive function in $C$. Then we have

$$0 = \int_D u(x) \psi(x) Lu(x) dx = \int_D a(x) \psi(x) u^2(x) dx - \int_D \sum_{i=1}^n u_i(x) u(x) \psi(x) dx,$$

where

$$u_{ik} = \frac{\partial^2 u}{\partial x_i \partial x_k}.$$

By partial integration we get

$$0 = \int_B M(u) ds + \int_D \sum_{i=1}^n u_i^2(x) \psi(x) dx +$$

$$+ \int_D \sum_{i=1}^n u_i(x) \psi_i(x) u(x) dx + \int_D a(x) \psi(x) u^2(x) dx,$$

(1)

where $M(u)$ contains $u$ and first derivatives of $u$ and where $ds$ denotes the surface element. In the third integral we can integrate by part once more and get

$$\int_D \sum_{i=1}^n u_i(x) \psi_i(x) u(x) dx = \int_B M'(u) ds - \frac{1}{2} \int_D \sum_{i=1}^n \psi_i(x) u_i^2(x) dx,$$

(2)

where $M'$ is an analogue of $M$. 282