A theorem of the Phragmén–Lindelöf type for second-order elliptic operators

By Lars Lithner

1. Introduction and notations

Let $\mathbb{R}^n$ be the real $n$-dimensional Euclidean space with coordinates $x = (x_1, x_2, \ldots, x_n)$, $|x| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$. $C$ denotes the set of all complex-valued infinitely differentiable functions on $\mathbb{R}^n$ with compact supports and $L^2(\Omega)$ is the Hilbert space of all complex-valued square integrable functions on the set $\Omega$.

In $\mathbb{R}^1$ let $D$ be the domain $\{x \mid x \geq a\}$ where $a$ is arbitrary and let $L$ be the differential operator

$$-\left(\frac{d}{dx}\right)^2 + \lambda, \lambda > 0.$$ 

The solutions of $Lu = 0$ are

$$u(x) = C_1 e^{\sqrt{\lambda} x} + C_2 e^{-\sqrt{\lambda} x},$$

where $C_1$ and $C_2$ are arbitrary constants. From this we conclude that if a solution is bounded in the domain $D$ or if it belongs to $L^2(D)$ then it decreases like $e^{-\sqrt{\lambda} x}$ when $x$ tends to infinity and the same holds for its derivative. In particular, $ue^{\mu x}$ and $(du/dx)e^{\mu x}$ belong to $L^2(D)$ if $\mu < \sqrt{\lambda}$.

In this paper we shall extend this result to second-order elliptic differential operators in $\mathbb{R}^n$.

$$L = -\sum_{i,k=1}^n a_{ik}(x) \frac{\partial^2}{\partial x_i \partial x_k} + \sum_{k=1}^n b_k(x) \frac{\partial}{\partial x_k} + a(x),$$

where $D_{ik} = \partial^2/\partial x_i \partial x_k$, $D_k = \partial/\partial x_k$ and $a_{ik}(x) = a_{ki}(x)$ (for simplicity we confine ourselves to the real domain).

Giving the result of the general case at the end of the paper we start with the operator $L = -\Delta + a(x)$, where $\Delta$ is the Laplace operator in $\mathbb{R}^n$ and where $a$ is positive and continuous or, more generally, locally bounded and Borel measurable. Then we can prove that if $u$ is a solution of $Lu = 0$ outside some compact set $K$ and if $u$ belongs to $L^2(\mathbb{R}^n - K)$ then, in the same sense as above, $u$ and its first derivatives decrease exponentially like $e^{-\varphi(x)}$ when $|x|$ tends to infinity. $\varphi(x)$ is the geodetic distance from the origin to the point $x$ in the metric $ds^2 = a(x)(dx_1^2 + dx_2^2 + \ldots + dx_n^2)$.
2. The special case

Let \( D \) be the domain \( \{ x \mid |x| \geq R \} \), where \( R \) is a positive number and let \( B \) be the boundary of \( D \). \( L \) is the operator \( -\Delta + a \) where the function \( a \) is strictly positive in \( \mathbb{R}^n \). Let \( \varphi(x) \) be the geodetic distance from the origin to the point \( x \) in the metric \( ds^2 = a(x)(dx_1^2 + dx_2^2 + \ldots + dx_n^2) \) that is, \( \varphi(x) \) is the greatest lower bound of

\[
\int_{\Gamma} a(y) \sqrt{dy_1^2 + dy_2^2 + \ldots + dy_n^2}, \quad y = (y_1, y_2, \ldots, y_n),
\]

where \( \Gamma \) is a piecewise continuously differentiable curve starting at the origin and ending at \( x \). Putting further conditions on \( \varphi \) will be continuously differentiable.

**Lemma 1.** \( |\text{grad} \varphi(x)| \leq \sqrt{a(x)} \).

**Proof.** It is evident from the definition of \( \varphi \) that

\[
|\varphi(x + \Delta x) - \varphi(x)| \leq \int_{\Gamma_0} \sqrt{a(y) \sqrt{dy_1^2 + dy_2^2 + \ldots + dy_n^2}},
\]

where \( \Gamma_0 \) is the straight line segment joining \( x \) and \( x + \Delta x \). This gives the inequality.

**Lemma 2.** (Carleman [1].) If \( u \) belongs to \( L^2(D) \) and is a solution of \( Lu = 0 \) then \( \sqrt{a} u \) and \( |\text{grad} u| \) belong to \( L^2(D) \).

**Proof.** Let \( \psi \) be a positive function in \( C \). Then we have

\[
0 = \int_D u(x)\psi(x)Lu(x)dx = \int_D a(x)\psi(x)u^2(x)dx - \sum_{i=1}^n \int_D u_i(x)u(x)\psi(x)dx,
\]

where

\[
u_{ik} = \frac{\partial^2 u}{\partial x_i \partial x_k}.
\]

By partial integration we get

\[
0 = \int_B M(u)ds + \sum_{i=1}^n \int_D u_i^2(x)\psi(x)dx + \sum_{i=1}^n u_i(x)u(x)\psi(x)dx + \int_D a(x)\psi(x)u^2(x)dx,
\]

where \( M(u) \) contains \( u \) and first derivatives of \( u \) and where \( ds \) denotes the surface element. In the third integral we can integrate by part once more and get

\[
\int_D \sum_{i=1}^n u_i(x)\psi_i(x)u(x)dx = \int_B M'(u)ds - \frac{1}{2} \sum_{i=1}^n \int_D \psi_{ii}(x)u^2(x)dx,
\]

where \( M' \) is an analogue of \( M \).