BRIEF COMMUNICATIONS

TWO THEOREMS IN THE THEORY OF SUMMATION OF NUMERICAL SERIES BY LOWER TRIANGULAR POSITIVE MONOTONE MATRICES

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For so-called monotone matrices, we establish necessary and sufficient conditions for the summation with them of some divergent sequences of 0 and 1 up to zero or some unbounded sequences of nonnegative numbers.

1. Mazur and Orlicz [1, p. 375] proved the following theorem: If a regular matrix sums up a bounded divergent sequence, then it sums up an unbounded sequence. However, this theorem does not contain conditions under which this matrix is boundedly efficient, i.e., it sums up at least one bounded divergent sequence.

In the present paper, we indicate the class of matrices boundedly and unboundedly efficient. Moreover, the Mazur–Orlicz theorem mentioned above is invertible on this class of matrices, although this theorem is not invertible on the class of all regular matrices.

2. A matrix \( \|a_{nk}\| \) is called monotone if it is lower triangular and satisfies the conditions

\[
\lim_{n \to \infty} a_{nk} = 0 \quad \forall k \quad \text{and} \quad |a_{n+1,k}| \leq |a_{nk}| \quad \forall k \in \{0, n, n \geq n_0\}. \tag{1}
\]

We note that a monotone matrix may be regular. But it is not necessary because conditions (1) contain only one of the three conditions of regularity.

The main results of the present paper are the following statements:

**Theorem 1.** In order that a monotone matrix \( \|a_{nk}\| \) sum up a certain divergent bounded nonnegative sequence \((s_n)\) (it is possible that \( s_n = 0 \) or \( s_n = 1 \) for any \( n \)) to zero, it is sufficient and, in the case of positive matrix, also necessary that

\[
\limsup_{n \to \infty} \|a_{nk}\| = 0. \tag{2}
\]

**Theorem 2.** In order that a monotone matrix \( \|a_{nk}\| \) sum up a certain unbounded nonnegative sequence \((s_n)\) (possibly, only with two partial limits 0 and \( +\infty \)) to zero, it is sufficient and, in the case of positive matrix, also necessary that equality (2) be satisfied.

**Corollary 1.** A monotone positive matrix \( \|a_{nk}\| \) sums up a certain divergent bounded nonnegative sequence to zero if and only if it sums up a certain unbounded nonnegative sequence to zero.
Corollary 2. Let \( z_0 \neq 0 \) and let \( F \) be a closed set. Moreover, let \( \{0, z_0\} \subset F \subset \{z : |z| \leq R\} \) (in particular, \( F = \{z : |z| \leq R\} \)). Then there exists a sequence of complex numbers \( (s_n) \) everywhere dense in \( F \), which is summed up to zero by a monotone matrix \( ||a_{nk}|| \) satisfying condition (2).

Corollary 3. The matrix of Cesaro means of order \( \alpha \geq 1 \) sums up certain divergent sequences from 0 and 1 to zero as well as certain unbounded nonnegative sequences, whose partial limits are only the numbers 0 and \( +\infty \).

Corollary 4. In order that the Riesz matrix \( (R, p_n) \), where \( p_0 > 0 \) and \( p_i \geq 0 \) for any \( i \),

\[
P_n = \sum_{i=0}^{n} p_i \to \infty, \quad n \to \infty,
\]

sum up a certain divergent sequence from 0 and 1 (a certain unbounded nonnegative sequence) to zero, it is necessary and sufficient that

\[
\lim \sup_{n \to \infty} \frac{P_n}{P_n} = 0.
\]  

(3)

Corollary 5. Suppose that the positive Voronoi matrix \( (W, p_n) \) satisfies the conditions

\[
p_0 \geq p_1 \geq \ldots \geq p_n \geq \ldots \text{ and } \lim_{n \to \infty} P_n = +\infty.
\]

Then this matrix sums up certain divergent sequences from 0 and 1 to zero as well as certain unbounded nonnegative sequences that have only two partial limits: 0 and \( +\infty \).

3. Proof of Theorems 1 and 2. Assume that a matrix \( ||a_{nk}|| \) is monotone and satisfies condition (2). We construct an increasing sequence \( (n_k) \) in the following way:

By virtue of conditions (1) and (2), there exists a number \( n_1 > 1 \) such that

\[
\sum_{i=0}^{1-1} |a_{n_in_i}| \leq 1, \quad |a_{n_1n_1}| \leq 1,
\]

where \( n_0 = 0 \).

Assume that we chose a number \( n_k \) such that the inequalities

\[
\sum_{i=0}^{k-1} |a_{n_in_i}| \leq \frac{1}{k}, \quad |a_{n_kn_k}| \leq \frac{1}{k}
\]  

(4)

are true.

Since \( \lim_{n \to \infty} a_{ni} = 0 \) for any \( i \), and \( \lim \sup_{n \to \infty} |a_{ni}| = 0 \), there exists a number \( n_{k+1} > n_k + 1 \) such that

\[
\sum_{i=1}^{k} |a_{n_{k+1}n_i}| \leq \frac{1}{k + 1}
\]