THE CONTINUOUS COLLAPSING KNAPSACK PROBLEM

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A Collapsing Knapsack is a container whose capacity diminishes as the number of items it must hold is increased. This paper focuses on those cases in which the decision variables are continuous, i.e., can take any non-negative value. It is demonstrated that the problem can be reduced to a set of two dimensional subproblems. Strategies for elimination of subproblems and conditions permitting reduction to a set of one dimensional problems are also considered.

Computational results indicate that the procedure is quite efficient. Even for large problems only a small number of subproblems have to be solved.

Key Words: Nonlinear, Knapsack.

1. Introduction

A Collapsing Knapsack is a container whose capacity diminishes as the number of items it must hold is increased. A computer operating in time sharing mode is a ready example. As the number of users on the system increases, a larger proportion of total operating time must be devoted to switching from one user to another. As a result, the throughput (capacity) of the system is reduced.

The problem was first addressed in [3]. The case where the decision variables were restricted to 0–1 was examined in [2] and [6], while the integer case was considered in [7]. Additional applications ranging from satellite communications to urban planning are also discussed in these papers. In contrast, this paper will consider the problem where the decision variables can take any non-negative value.

Formally, the problem may be stated as

\[
\begin{align*}
\text{maximize} & \quad z(x) = \sum_{i=1}^{n} c_i x_i, \\
\text{subject to} & \quad \sum_{i=1}^{n} k_i x_i \leq h\left(\sum_{i=1}^{n} x_i\right), \\
& \quad x \geq 0,
\end{align*}
\]

(P1)

where row vectors \( c, k > 0 \) are given, and \( h \) is a left continuous nonincreasing functional with \( 0 \leq h(0) < \infty \).

More concretely, consider the transportation of goods where fuel capacity is a scarce resource. This would be the case, for example, with air or rocket

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transport. As the size of the shipment increases, the space available for fuel decreases, and trade-offs must be made between amount of fuel and amount of cargo to be carried. Therefore, let $x_i$ be the number of liters of good $i$ chosen, $c_i$ the value and $k_i$ the weight of one liter of $i$. Then $\sum k_i x_i$ represents the weight of a given shipment and $\sum x_i$ is its total size. The maximum allowable weight is $h(y)$ given that $y$ of the available space has been used for cargo.

A non-integral vector $x$ is helpful to model goods which are 'infinitely' divisible, such as liquids or raw materials. However, even where $x$ is necessarily an integer, knowledge of (P1) and its structure may provide useful bounds, approximate solutions, and insights into the problem in question.

As has been demonstrated (see [1], e.g.), the classical continuous knapsack problem,

$$\text{maximize } z_2(x) = cx,$$

subject to

$$kx \leq b, \quad x \geq 0,$$

where $c$ and $k$ are $n$-dimensional row vectors, $x$ is a column vector, and $b$ is a scalar, has an optimal solution with at most one variable positive. It can be shown that the continuous collapsing knapsack problem (P1) has an optimal solution with at most two variables positive. Thus (P1) can be decomposed into a set of $\binom{n}{2}$ two dimensional subproblems. Analysis of this structure reveals that at most $n - 1$ of these subproblems need to be solved. Strategies for the further elimination of subproblems, and limitations on (P1) which insure a solution with only one variable positive will be considered. Finally, an algorithmic procedure and computational results will be presented.

2. Problem structure

We begin by demonstrating that the restrictions which are placed on $h$ in problem (P1) are sufficient to guarantee an optimal solution.

Theorem 2.1. (P1) has a finite optimal solution.

Proof. Since $k \cdot 0 \leq h(0)$, $x = 0$ is a trivial feasible solution to (P1) and the feasible solution set is not empty. Because $h$ is non-increasing, left continuity implies that $h$ is upper semi-continuous. Therefore, any sequence of feasible points with a limit point has its limit in the feasible region. In other words, the feasible solution set is closed.

Since $h(0) < \infty$, there exists $M \geq h(y)$ for all $y > 0$. And $k_i x_i \leq \sum k_i x_i \leq h(\sum x_i) \leq M$. Thus $x_i \leq M/k_i < \infty$. The feasible region is bounded and hence compact. Weierstrass' theorem, which states that a continuous function over a compact set attains its maximum, completes the proof.