AN APPLICATION OF HOMOTOPY TO SOLVING LINEAR PROGRAMS

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Consider a linear program in \( m \) inequality constraints and \( n \) nonnegative variables. An application of homotopy to the problem gives an algorithm similar to Dantzig’s self-dual method. However, the homotopy approach allows one to recognize several previously undescribed and potentially interesting properties. For example, the algorithm can be initiated in such a way as to produce a path which is primal–dual feasible. Moreover, one can theoretically identify an orthant with the property that if one initiates the algorithm at any point in that orthant then, after a ‘phase I’ requiring at most \( \min(m, n) \) pivots, convergence is obtained in one step.

Key words: Homotopy, Self-Dual Method, Simplex Method, Linear Programming, Linear Complementarity.

1. Introduction

In this paper, a homotopy approach to solving linear programming problems is described. This leads to a path in the primal–dual space. Under certain specific choices of the starting point, and viewed in an appropriate (different from the original primal–dual) space, the algorithm becomes an application of other known methods such as Lemke’s and, more relevantly, Dantzig’s self-dual method [3]. Some properties of the procedure are:

(1) It solves the primal and dual problems at the same time.

(2) It follows a piecewise-linear path in each of the primal and dual spaces, and not the edges of the primal or dual polyhedra.

(3) It always converges if an optimal solution exists.

(4) In general, neither primal nor dual objective function is monotonic in improvement.

(5) The scattering point is some specified vector \( z^0 = (x^0, y^0) \) in primal–dual space and there is considerable flexibility in its choice:

(a) If \( z^0 < 0 \), no ‘phase I’ is required and the algorithm is ‘outside-in’, i.e. the path is primal infeasible and dual infeasible.

(b) If \( z^0 \geq 0 \) and \( z^0 \) strictly satisfies all other constraints (this may require some work), the algorithm is ‘inside-out’, i.e. the path remains (primal and dual) feasible

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at each step. Thus, at each step, the primal and dual objective values give a bound on optimality.

(c) If the primal and dual problems have unique solutions \( x^*, y^* \) (i.e. the primal and dual problems are nondegenerate) then there exists an orthant, say \( E_\star \), such that if \( z^0 \in E_\star \), the algorithm converges in one step, after a 'phase I' which requires at most \( \min\{m, n\} \) steps, where the primal problem is \( m \) inequality constraints in \( n \) nonnegative variables.

(6) The algorithm operates ( pivots ) on a tableau which has one more row and one more column than the tableau employed in the usual version of the simplex method.

Although various aspects of the material herein described reduce to known procedures, the general insights obtained via the homotopy structure lead to several previously unknown features ( e.g. 5(b) and 5(c) above ). At the present time one can express only a guarded assessment, but it is possible that these new features may have practical importance.

To begin, consider the pair of dual linear programming problems:

\[
\text{(P) } \max c^T x \quad \text{(D) } \min b^T y
\]

\[
s.t. \quad Ax \leq b, \quad s.t. \quad A^T y \geq c,
\]

\[
x \geq 0; \quad y \geq 0;
\]

here \( A \in \mathbb{R}^{m \times n} \) and the remaining entries have appropriate dimensions. As is well known [3] either of these problems has an optimal solution if and only if the other does also. A necessary and sufficient condition for \( x^* \) to be optimal in (P), and \( y^* \) to be optimal in (D), is that \( x^*, y^* \) satisfy the three conditions

(i) \[
\begin{bmatrix}
-c \\
b
\end{bmatrix} + \begin{bmatrix}
0 & A^T \\
-A & 0
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix} \geq 0,
\]

(ii) \[
\begin{bmatrix}
x \\
y
\end{bmatrix} \geq 0,
\]

(iii) \[
[x^T, y^T] \cdot \left\{ \begin{bmatrix}
-c \\
b
\end{bmatrix} + \begin{bmatrix}
0 & A^T \\
-A & 0
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix} \right\} = 0.
\]

Define

\[
q = \begin{bmatrix}
-c \\
b
\end{bmatrix}, \quad M = \begin{bmatrix}
0 & A^T \\
-A & 0
\end{bmatrix}, \quad z = \begin{bmatrix}
x \\
y
\end{bmatrix}.
\]

Then with these identifications the above optimality conditions have the form of the linear complementarity problem

\[
\text{(LCP) } \quad \text{find } z \geq 0 \text{ such that } q + Mz \geq 0 \text{ and } z^T(q + Mz) = 0.
\]

\footnote{As a historical note, it should be stated that the reduction of (P) and (D) to an equivalent (LCP), and then solving the latter with Lemke's method, is a well-known technique, and is, for example, discussed in [7, 8, 9, 10]. Van de Panne states [10, p. 427]: "For linear programming, usage of the format of the linear complementarity problem doubles the size of the problem without any advantage". Ravindran [8] has given a method for following Lemke's path with a compact tableau.}