ON THE LENGTH OF SIMPLEX PATHS:
THE ASSIGNMENT CASE

P.O. LINDBERG
Department of Mathematics, Royal Institute of Technology, Stockholm, Sweden

Snjólfur ÓLAFSSON
Science Institute, University of Iceland, Dunhaga 3, 107 Reykjavik, Iceland

Received 6 July 1981
Revised manuscript received 12 December 1983

This paper reports on an experimental study of the distribution of the length of simplex paths for the Optimal Assignment Problem. We study the distribution of the pivot counts for a version of the simplex method that with essentially equal probabilities introduces any variable with negative reduced cost into the basis.

In this situation the distribution of the pivot counts turns out to be normally distributed and independent of the actual cost coefficients, provided these are sufficiently spread out. Further, the mean and standard deviation grow only moderately with the size of the problem, namely as $d^{1.8}$ and $d^{1.5}$ respectively for a $d \times d$ problem, implying in particular that the pivot counts concentrate around the mean with growing $d$.

The usual simplex method on the other hand gives a growth of $d^{1.6}$. Hence a large part of the favourable polynomial growth experienced on practical problems may be attributed to the fact that the simplex paths are rather short on the average, at least for assignment problems.

Key words: Linear Programming, Simplex paths, Number of pivots, Assignment Problems, Polynomial Growth.

1. Introduction and overview

This paper reports on an experimental study of the distribution of simplex path lengths for the Optimal Assignment Problem. (With a simplex path we understand a sequence of pivots from a given basis to the optimal one where at each pivot some variable with negative reduced cost is introduced into the basis.)

One may say that the paper studies a stochastic version of the simplex method: at each pivot all variables with negative reduced cost have equal probability of entering the basis. We then study the distribution of the iteration counts for this method, 'the stochastic simplex method', for a fixed problem as depending on the problem size.

Our main result is that for a given (suitably standardized) $d \times d$ assignment problem the pivot count for the stochastic simplex method (slightly modified for computational reasons) is approximately normal with mean and standard deviation proportional to $d^{1.8}$ and $d^{1.5}$ respectively. These results have a very strong statistical support in that there is very little lack of fit in the performed regressions, showing
that the mean iteration count almost entirely depends on the dimension and not on
the specific problem at hand. This indicates that the stochastic simplex method might
be polynomial with probability one. For the standard simplex method in the form
of PNET (Glover et al. (1974)), the iteration count is seen to grow approximately as \(d^{1.6}\).

It was (and is) our belief that a large part of the efficiency of the simplex method
may be explained by the existence of a large number of fairly short simplex paths.
In an endeavour to experimentally study the distribution of simplex path lengths,
we started with assignment problems because these are easy to generate and
standardize.

To our knowledge there is no paper treating the efficiency of the simplex method
on assignment problems. For the general linear programming problem, however,
there are several papers. In Section 2 we give a short review of this area and also
explain the origin of our ideas.

The (modified) stochastic simplex method and the generation and solution of our
assignment problems are described in Section 3.

For a given problem dimension we typically experiment on only one assignment
problem. For the results of different dimensions to be comparable, the selected
problems have to be standardized in some way. This is studied in Section 4, where
we show that, provided the cost coefficients are sufficiently ‘spread out’, then the
distribution of the iteration counts only depends on the dimension of the problem
and that with less variation in the cost coefficients we only get lower pivot counts.
Further the iteration counts are seen to be approximately normal.

In Section 5 we describe the setup of the regression analysis which is performed
in Section 6 and Section 7 for the variance \(\sigma_d^2\) and the mean \(\mu_d\) of the iteration
counts. It is shown that the models

\[
\mu_d = 1.19 \ d^{1.83} \quad \text{and} \quad \sigma_d = 0.33 \ d^{1.51}
\]

explain practically all the variation in the data for \(d\) between 20 and 230.

In Section 8 we compare our method with the standard simplex method, for
which we show that the iteration count grows only moderately slower, namely as
\(1.10 \ d^{1.57}\).

In Section 9 we compare the stochastic simplex method with our modification of
it. It seems that the difference in distribution for the two methods is small and
decreases with increasing dimension.

In Section 10 we try to draw some conclusions from our study.

This paper is based on the report Lindberg and Ólafsson (1980). However, it
contains results for a wider range of dimensions than that report and is using other
data.

The strong results in the paper motivate some rather careful tests of normality,
problem independence etc. The details of these tests may be found in our earlier
report.

As general reference to statistical theory we will use Dixon and Massey (1957).