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TWO ALGORITHMS FOR WEIGHTED MATROID INTERSECTION

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Consider a finite set E, a weight function w: E \to \mathbb{R}, and two matroids M_1 and M_2 defined on E. The weighted matroid intersection problem consists of finding a set I \subseteq E, independent in both matroids, that maximizes \( \Sigma \{ w(e) : e \in I \} \). We present an algorithm of complexity \( O(nr(r+c+\log n)) \) for this problem, where \( n = |E|, \ r = \min(\text{rank}(M_1), \text{rank}(M_2)), \ c = \max(c_1, c_2) \) and, for \( i = 1, 2, \) \( c_i \) is the complexity of finding the circuit of \( I \cup \{ e \} \) in \( M_i \) (or show that none exists) where \( e \) is in \( E \) and \( I \subseteq E \) is independent in \( M_1 \) and \( M_2 \). A related problem is to find a maximum weight set, independent in both matroids, and of given cardinality \( k \) (if one exists). Our algorithm also solves this problem. In addition, we present a second algorithm that, given a feasible solution of cardinality \( k \), finds an optimal one of the same cardinality. A sensitivity analysis on the weights is easy to perform using this approach. Our two algorithms are related to existing algorithms. In fact, our framework provides new simple proofs of their validity. Other contributions of this paper are the existence of nonnegative reduced weights (Theorem 6), allowing the improved complexity bound, and the introduction of artificial elements, allowing an improved start and flexibility in the implementation of the algorithms.

Key words: Matroids, matroid intersection algorithm, polynomial algorithm.

1. Introduction

Let \( E \) be a finite set, \( M_1 \) and \( M_2 \) two matroids defined on \( E \), and \( w: E \to \mathbb{R} \) a weight function. The weight of a subset \( F \subseteq E \) is defined by \( w(F) = \Sigma \{ w(e) : e \in F \} \). The weighted matroid intersection problem consists of finding a maximum weight set \( I \subseteq E \) that is independent in both matroids.

Let \( k \) be a positive integer. Call \( F \subseteq E \) a \( k \)-intersection if \( |F| = k \) and \( F \) is independent in both \( M_1 \) and \( M_2 \). We also consider the problem of finding a maximum weight \( k \)-intersection \( I \subseteq E \), if one exists. This problem is denoted by \( (P_k) \).

Problem \( (P_k) \) and the weighted matroid intersection problem have been solved in polynomial time by Edmonds (1970, 1979), Lawler (1976), Iri and Tomizawa (1976), Frank (1981), and Orlin and Vande Vate (1984). Their algorithms start with \( I = \emptyset \) and increase the cardinality of \( I \) by one at each iteration until an optimum \( k \)-intersection is found.

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We discuss a different algorithm for \( (P_k) \), one that operates in a "horizontal" fashion. This algorithm is closely related to the primal algorithm of Fujishige (1977). It starts with a \( k \)-intersection \( I \). The set \( I \) is used to construct a digraph related to Glover's state graphs (1985) and Lawler's border graphs (1976). We then use negative length dicycles in our digraph to find an improved \( k \)-intersection. The procedure is repeated until an optimum set is found. In constructing our digraph from \( I \) at each iteration, the following circuit recognition problem arises in matroids \( M_1 \) and \( M_2 \). Given a set \( I \subseteq E \), independent in both matroids, and \( e \) in \( E - I \), find the circuit of \( I + e \) in \( M_i \) (or show that none exists) for \( i = 1, 2 \). We denote by \( c \) the complexity of this circuit recognition problem. Note that we use summation notation to represent set union.

This "horizontal" approach, in conjunction with the introduction of artificial elements, yields a second algorithm for problem \( (P_k) \). It has complexity \( O(nk(k + c + \log n)) \), where \( n = |E| \). Our second algorithm also solves the weighted matroid intersection problem. It is closely related to Lawler's primal algorithm, the main difference being that, at each iteration, it only requires finding a shortest dipath between two prespecified nodes in a digraph with nonnegative arc weights and therefore does not need the more time-consuming labeling procedure used by Lawler. This accounts for the term \( nk(k + \log n) \) in our complexity bound instead of Lawler's \( n^2k^2 \).

Our development uses concepts and results from Glover's paper on the generalized quasi-greedy algorithm (1985). We include them here to make this report self-contained.

2. Preliminary results

Let \( M_1 \) and \( M_2 \) be two matroids defined on the same element set \( E \). Throughout this section we assume that \( I \) is independent in both matroids. Let \( \Sigma \) be a subset of \( I \) and let \( \Sigma' \) be a subset of \( E - I \) such that \( |\Sigma| = |\Sigma'| \). We say that \( (\Sigma, \Sigma') \) is an \( I \) swap if \( I - \Sigma + \Sigma' \) is independent in \( M_1 \) and an \( I \) back-swap if \( I - \Sigma + \Sigma' \) is independent in \( M_2 \). Also, we call \( m(\Sigma, \Sigma') \) a matching if it represents a one-one mapping of \( \Sigma \) onto \( \Sigma' \); \( m(\Sigma, \Sigma') \) is called an \( I \) matching if every \( (e, e') \) in \( m(\Sigma, \Sigma') \) is an \( I \) swap, and an \( I \) back-matching if every \( (e, e') \) in \( m(\Sigma, \Sigma') \) is an \( I \) back-swap.

**Lemma 1.** Let \( I \) and \( I' \) be independent sets in a matroid \( M \). Let \( \Sigma = I - I' \) and \( \Sigma' = I' - I \). Choose any \( e \) in \( \Sigma \) such that \( I' + e \) is dependent. Then there is an \( e' \) in \( \Sigma' \) such that both \( I' + e - e' \) and \( I + e' - e \) are independent in \( M \).

**Proof.** Since \( I' + e \) is dependent, it contains a unique circuit \( I'(e) \). Clearly, every \( e' \) in \( I'(e) - e \) will satisfy the requirement that \( I' + e - e' \) be independent. We need to show that at least one such \( e' \) yields \( I + e' - e \) independent.