ASYMPTOTIC SOLUTION OF THE CAUCHY PROBLEM
FOR A SINGULARLY PERTURBED LINEAR SYSTEM

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We construct the asymptotics of the solution of the Cauchy problem for a degenerate singularly perturbed linear system in the case of multiple spectrum of the principal operator.

Consider the Cauchy problem

\[ \varepsilon B(t) \frac{dx}{dt} = A(t, \varepsilon)x + f(t, \varepsilon), \]
\[ x(0, \varepsilon) = x_0. \]

where \( t \in [0, T] \), \( B(t) \) and \( A(t, \varepsilon) \) are \((n \times n)\) matrices, \( x(t, \varepsilon) \) and \( f(t, \varepsilon) \) are \( n \)-dimensional vectors, \( \varepsilon > 0 \) is a small real parameter, \( h \in \mathbb{N} \), \( \det B(t) = 0 \) on \([0, T]\), and \( B(t) \in C_0^{\infty} \).

Assume that the following conditions are satisfied:

(i) the matrix \( A(t, \varepsilon) \) and the vector \( f(t, \varepsilon) \) on the given segment \([0, T]\) allow for uniform developments in powers of the small parameter \( \varepsilon \):

\[ A(t, \varepsilon) \sim \sum_{k \geq 0} \varepsilon^k A_k(t), \quad f(t, \varepsilon) \sim \sum_{k \geq 0} \varepsilon^k f_k(t), \quad \text{where} \quad A_k(t), f_k(t) \in C_0^{\infty}; \]

(ii) the pencil of matrices \( L(t, \lambda) = A_0(t) - \lambda B(t) \) is regular on \([0, T]\) and has one finite elementary divisor \((\lambda - \lambda_0)^p\) of multiplicity \( p \) and one infinite elementary divisor of multiplicity \( q = n - p \).

A solution of problem (1), (2) will be constructed in the form

\[ x(t, \varepsilon) = \sum_{i=1}^{p} u_i(t, \varepsilon) \exp \left( \varepsilon^{-h} \int_{0}^{t} (\lambda_0 + \lambda_i(t, \varepsilon)) \, dt \right) + \sum_{j=1}^{q-1} v_j(t, \varepsilon) \exp \left( \varepsilon^{-h} \int_{0}^{t} \frac{dt}{\xi_j(t, \varepsilon)} \right) + w(t, \varepsilon), \]

where the \( n \)-dimensional vector functions \( u_i(t, \varepsilon) \), \( i = \overline{1, p} \), and \( v_j(t, \varepsilon) \), \( j = \overline{1, q-1} \), and the scalar functions \( \lambda_i(t, \varepsilon) \), \( i = \overline{1, p} \), and \( \xi_j(t, \varepsilon) \), \( j = \overline{1, q-1} \), are represented by formal developments in powers of \( \mu = \sqrt[\varepsilon]{\varepsilon} \) and \( \nu = q^{-\sqrt[\varepsilon]{\varepsilon}} \), and the vector function \( w(t, \varepsilon) \) is represented by a formal development in powers of \( \varepsilon \):

\[ u_i(t, \varepsilon) = \mu^{-(p-1)} \sum_{k=0}^{\infty} \mu^k u_k^{(i)}(t), \quad \lambda_i(t, \varepsilon) = \sum_{k=1}^{\infty} \mu^k \lambda_k^{(i)}(t), \quad i = \overline{1, p}. \]
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\( v_j(t, \varepsilon) = \varepsilon^{-(q-2)} \sum_{k=0}^{\infty} \varepsilon^k v_k^{(j)}(t), \quad \xi_j(t, \varepsilon) = \sum_{k=1}^{\infty} \varepsilon^k \xi_k^{(j)}(t), \quad j = \overline{1, q}, \)

\( w(t, \varepsilon) = \varepsilon^{-m} \sum_{k=0}^{\infty} \varepsilon^k w_k(t) \)

\((m = 0\) in the “nonresonance” case, in which the eigenvalue of the pencil \( L(t, \lambda) \) is different from zero, and \( m = -1 \) in the case of “resonance,” in which this eigenvalue is equal to zero).

Finding of the coefficients of developments (4) is performed by substituting (3), (4) into (1), (2) and solving the system of algebraic equations formed after equating coefficients at similar exponents and similar powers of the parameters:

\( (A_0 - \lambda_0 B) u_k^{(i)} = b_k^{(i)} = \sum_{s=1}^{k} \lambda_s^{(i)} B u_k^{(i)} - s + B(u_k^{(i)} - \lambda_0 B) u_k^{(i)} - s - \sum_{s=1}^{[k/p]} A_s u_k^{(i)} - s, \)

\( i = \overline{1, p}, \)

\( B u_k^{(j)} = a_k^{(j)} = \sum_{s=0}^{k - h(q - 1)} \lambda_s^{(j)} B u_k^{(j)} - s + B(u_k^{(j)} - \lambda_0 B) u_k^{(j)} - s - \sum_{s=1}^{k - h(q - 1)} B \xi_k^{(j)}(v_k^{(j)} - s - h(q - 1))', \quad j = \overline{1, q - 1}, \)

\( A_0 w_k = d_k = B(w_k - \lambda_0) - s + \sum_{s=1}^{k - h(q - 1)} B \xi_k^{(j)}(v_k^{(j)} - s - h(q - 1))', \quad j = \overline{1, q - 1}, \)

\( \sum_{i=1}^{p} u_k^{(i)}(0) + \sum_{j=1}^{q-1} v_k^{(j)}(0) + w_k^{(i)}(0) + w_k(0) = \delta_{k, (p - 1)(q - 1)} x_0, \)

where, by definition, we set \( v_{m/p} = 0 \) if \( m \) is not divisible by \( p \), and \( \delta_{k,p} \) is the Kronecker symbol.

Systems (5)–(7) are solved by the method presented in [1].

The vector functions \( u_k^{(i)}(t) \) and \( v_k^{(j)}(t) \) are defined by the formulas

\( u_k^{(i)}(t) = H(t) b_k^{(i)}(t) + C_k^{(i)} \phi(t), \quad v_k^{(j)}(t) = G(t) a_k^{(j)}(t) + \tilde{C}_k^{(j)} \tilde{\phi}(t), \)

\( i = \overline{1, p}; \quad j = \overline{1, q - 1}; \quad k = 0, 1, \ldots, \)

in which \( \phi(t) \) is the eigenvector of the pencil \( L(t, \lambda) \), \( \tilde{\phi}(t) \) is the eigenvector of the matrix \( B(t) \) that corresponds to its zero eigenvalue, \( H(t) \) and \( G(t) \) are seminvers of the matrices \( L(t, \lambda_0) \) and \( B(t) \), respectively, and \( C_k^{(i)} \) and \( \tilde{C}_k^{(j)} \) are constant scalar factors. In order to find the functions \( \lambda_k^{(i)}(t) \) and \( \xi_k^{(j)}(t) \), one uses the conditions of compatibility of systems (5), (6): \( \langle b_k^{(i)}, \psi \rangle = 0; \quad \langle a_k^{(j)}(t), \tilde{\psi} \rangle = 0, \) where \( \psi(t) \) and \( \tilde{\psi}(t) \) are the eigenvectors, respectively, of the matrix pencil conjugate to \( L(t, \lambda) \) and of the matrix conjugate to \( B(t) \).

For this iteration process to work, it is necessary and sufficient that the following conditions be satisfied: