We establish a necessary and sufficient condition of conjugacy of \( m \)-functions on surfaces.

This work follows the line started in 1986 by the works of Fomenko on topological classification of Hamiltonian systems and then continued in his works together with Matveev, Zieschang, Bolsinov, Brailov, Oshemkov, Trofimov, Sharko, and others [1 – 3]. Its subsequent development is summarized in [4]. Soon after that, this theory was applied also in other areas of mathematics, for example, in the Morse theory and in the theory of Morse–Smale vector fields [4 – 9]. In the present paper, we consider the question of equivalence of \( m \)-functions on surfaces.

### 1. The Notion of an \( m \)-Function on a Surface

Let \( M \) be a surface with boundary \( \partial M = M_1 \cup M_2 \cup M_0 \), where \( M_i, \ i = 0, 1, 2, \) are one-dimensional submanifolds (segments and circles) such that \( M_1 \cap M_2 = \emptyset \) and each of the sets \( M_{01} = M_1 \cap M_0 \) and \( M_{02} = M_2 \cap M_0 \) is either empty or consists of a finite number of points.

**Definition 1.** The quadruple \( \xi = (M; M_0, M_1, M_2) \) is called a cobordism with angles.

**Definition 2.** A smooth function \( f: M \to [a, b] \) is called an \( m \)-function on a cobordism with angles \( \xi \) if

(i) \( f \) has only finitely many critical points, all of which are nondegenerate and lie in the interior of \( M \);

(ii) \( M_1 = f^{-1}(a) \) and \( M_2 = f^{-1}(b) \);

(iii) the restriction \( f|_{M_0} \) is a Morse function on the cobordism \( (M_0; M_{01}, M_{02}) \).

**Remark.** Henceforth, when speaking about an \( m \)-function on a surface (possibly with empty boundary), we will always imply that some representation of this surface is specified in the form of a cobordism with angles; in this case, for different functions this representation may be different. If, in this case, \( f, g : M \to R \) are \( m \)-functions, then the corresponding cobordisms with angles will be denoted by

\[
\xi^f = (M; M_0^f, M_1^f, M_2^f) \quad \text{and} \quad \xi^g = (M; M_0^g, M_1^g, M_2^g).
\]

For every subset \( K \subseteq R \), we set \( C_K = f^{-1}(K) \) and \( D_K = g^{-1}(K) \).

Let \( f, g : M \to R \) be two \( m \)-functions on a surface \( M \). They will be said to be equivalent if there exist diffeomorphisms \( h: M \to M \) and \( \varphi: R \to R \), with \( \varphi \) preserving the orientation of \( R \), such that the relation \( \varphi \circ f = g \circ h \) is satisfied.

\( m \)-Functions have the following types of singular points: If \( x \in \text{Int} M \) is a nondegenerate critical point of the restriction of a function \( f \) on \( \text{Int} M \), then the definition of its index is the usual one. If again \( x \in \partial M \) is a critical point of the restriction \( f|_{\partial M} \), then its index is the pair \( (\lambda, \varepsilon) \), where \( \lambda \) is the index of the point \( x \) as of a...
nondegenerate critical point of \( f|_{\partial M} \), \( \varepsilon = +1 \) if the vector \( \text{grad}_x f \) is directed outward from \( M \), and \( \varepsilon = -1 \) if the vector \( \text{grad}_x f \) is directed inward toward the surface \( M \). Thus, there exist precisely seven types of singular points of \( m \)-functions, namely, critical points of indices 0, 1, and 2 in the interior of \( M \) and critical points of indices \( (0, \pm 1) \) and \( (1, \pm 1) \) of the restriction \( f|_{\partial M} \) of the function \( f \) on the boundary \( \partial M \).

In [5], a classification of Morse functions is given. We use the approach presented therein for classification of a broader class of functions.

The idea of the classification is that each \( m \)-function is canonically put into a correspondence with some graph with additional information (notation), called a \textit{molecule}. The name is explained by the fact that, for its construction, one introduces more simple graphs, called \textit{elementary particles}, from which one constructs what are called \textit{atoms}, from which, in turn, a molecule is constructed. Elementary particles describe the behavior of an \( m \)-function in a neighborhood of a critical point, atoms describe its behavior on one critical level, and the molecule describes the \( m \)-function as a whole. The main result of this paper is the following theorem:

**Theorem 1.** Two \( m \)-functions \( f \) and \( g \) defined on a compact surface \( M \) are equivalent if and only if the corresponding molecules are isomorphic.

2. Auxiliary Statements

Let \( f \) be an \( m \)-function on a surface \( M \) and let \( \xi = (M; M_0, M_1, M_2) \) be a representation of \( M \) in the form of a cobordism with angles. Furthermore, let \( K \subseteq M_0 \) be a compact subset, and let \( W \) be an open neighborhood of the set \( K \) in \( M \) that does not contain singular points of the \( m \)-function \( f \). Finally, let \( \Omega \) be an arbitrary gradient-like vector field of the function \( f \).

**Lemma 1.** There exists a gradient-like vector field \( \Sigma \) of the function \( f \) that coincides with \( \Omega \) on \( M \setminus W \) and is tangent to the boundary \( \partial M \) at the points of the set \( K \).

**Proof.** To prove the lemma, we use the standard technique, namely, we construct some open covering \( \{W_i\} \) of the manifold \( M \) and, on every element \( W_i \) of it, we define a gradient-like vector field \( \Sigma_i \) of the function \( f|_{W_i} \) tangent to the boundary at the points of the set \( K \cap W_i \). By gluing these fields together with the use of a partition of unity, we obtain the desired field \( \Sigma \). It is tangent to the boundary at the points of the set \( K \), being a linear combination of fields tangent to the boundary.

**Corollary 1.** Let \( f \) be an \( m \)-function on a cobordism \( \xi \) such that its restriction \( f|_{\partial M} \) on the boundary does not have critical points. Then there exists a gradient-like vector field of the function \( f \) on \( M \) tangent to the boundary at all the points of the set \( M_0 \).

**Proposition 1.** Two \( m \)-functions \( f \) and \( g \) without singular points on a surface \( M \) are equivalent.

**Proof.** Using Corollary 1, we construct gradient-like vector fields of the functions \( f \) and \( g \) tangent to \( M_0^f \) and \( M_0^g \), respectively. Now Proposition 1 follows from reasoning similar to that used in Theorem 3.4 of [10].

3. Singular Points of \( m \)-Functions on Surfaces and Their Elementary Particles

In defining elementary particles, we follow [5].

Let \( f \) be an \( m \)-function defined on an open subset \( U \) of some half-space in \( R^2 \) that has a unique singular point \( p \) in \( U \). We also assume that, in given local coordinates, this function has a canonical representation given either by the Morse lemma [11] or by a similar lemma for nondegenerate critical points [12] (Lemma 3.1). Lemma