ON THE GLOBAL CONVERGENCE OF PATH-FOLLOWING METHODS TO DETERMINE ALL SOLUTIONS TO A SYSTEM OF NONLINEAR EQUATIONS

Immo DIENER

Institut für Numerische und Angewandte Mathematik, Georg-August Universität Göttingen, D-3400 Göttingen, FR Germany

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In this paper we prove a general theorem stating a sufficient condition for the inverse image of a point under a continuously differentiable map from \( \mathbb{R}^n \) to \( \mathbb{R}^k \) to be connected. This result is applied to the trajectories generated by the Newton flow. Several examples demonstrate the applicability of the results to nontrivial problems.

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1. Introduction

We consider the problem of determining \( Z(F) \), the set of all zeros of a continuously differentiable map \( F: \mathbb{R}^n \to \mathbb{R}^n \). Only during the last decade has this problem been investigated in the literature and still there are only a few papers on this subject. Most of these papers describe various methods which might in some cases yield success, but results concerning the above problem have only been obtained for certain classes of polynomial-like functions (cf. [5, 6]). Among the path-following algorithms there are basically two distinct approaches, namely homotopy methods and trajectory methods, but various connections exist between these two methods [4].

A procedure that finds all solutions to a system of nonlinear equations has numerous applications for instance in physics, in engineering and in economics. Also, with such a procedure, the global optimum of a scalar function can be found by considering the gradient equations and by determining all stationary points. In this paper we prove a general theorem, which can be applied to several path-following methods to yield global convergence results. Some examples are given in Section 3.

2. A global convergence theorem

The following theorem states sufficient conditions, such that the inverse images of points under a map \( F: \mathbb{R}^n \to \mathbb{R}^k \), \( k \leq n \), are all connected \((n-k)\)-dimensional.

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submanifolds of $\mathbb{R}^n$. The idea is to construct a retraction from $\mathbb{R}^n$ onto $F^{-1}(0)$. The conditions are rather restrictive but, however, are satisfied in a number of cases. Some examples are given in Section 3.

**Theorem 2.1.** For $k \leq n$ let $F$ be a function in $C^2(\mathbb{R}^n, \mathbb{R}^k)$ such that

$$\sup \{ \| (DF(x)D^T F(x))^{-1} \| \, | x \in \mathbb{R}^n \} \leq K < \infty.$$ 

Then $F^{-1}(0)$ is a connected $(n-k)$-dimensional submanifold in $\mathbb{R}^n$.

**Proof.** By the regularity condition, $F^{-1}(0)$ is trivially a $(n-k)$-submanifold in $\mathbb{R}^n$. Only the connectedness needs to be proved. Consider the $C^1$-vectorfield on $\mathbb{R}^n$, given by

$$V : x \mapsto -D^T F(x)(DF(x)D^T F(x))^{-1} \frac{F(x)}{1 + \| F(x) \|}.$$ 

For all $x \in \mathbb{R}^n$ we have (with $\cdot$ denoting the euclidean norm)

$$\| V(x) \| \leq \frac{\| F(x) \|}{\sqrt{1 + \| F(x) \|}} < \sqrt{K}.$$ 

Thus the vectorfield is defined on the whole $\mathbb{R}^n$ and bounded. It follows that $V$ determines a global $C^1$ flow $\Phi(t, x)$ such that

$$\frac{\partial \Phi(t, x)}{\partial t} = V(\Phi(t, x)) \quad \text{and} \quad \Phi(0, x) = x.$$ 

Therefore we have

$$\frac{\partial}{\partial t} F(\Phi(t, x)) = - \frac{F(\Phi(t, x))}{1 + \| F(\Phi(t, x)) \|}$$

and

$$\frac{\partial}{\partial t} \| F(\Phi(t, x)) \|^2 = -2 \frac{\| F(\Phi(t, x)) \|^2}{1 + \| F(\Phi(t, x)) \|} \leq 0.$$ 

Thus for any $x_0 \in \mathbb{R}^n$ and any $t \geq 0$:

$$\| F(\Phi(t, x_0)) \|^2 = \| F(x_0) \|^2 \exp \left( -2 \int_0^t \frac{d\tau}{1 + \| F(\Phi(\tau, x_0)) \|} \right)$$

$$\leq \| F(x_0) \|^2 \exp \left( -2t \frac{1}{1 + \| F(x_0) \|} \right).$$

It follows that

$$\lim_{t \to \infty} \| F(\Phi(t, x_0)) \| = 0.$$