The Multivariate Extremal Index and the Dependence Structure of a Multivariate Extreme Value Distribution

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Abstract

Let \( H \) be the limiting distribution of a vector of maxima from a \( d \)-dimensional stationary sequence with multivariate extremal index. We give necessary and sufficient conditions for \( H \) to have independent or totally dependent margins by using relations between the multivariate extremal index and the univariate extremal indexes.

A new functional family of multivariate extreme value distributions, containing \( H \), is introduced. We apply the results to characterize the asymptotic independence of the maximum and the minimum and compute the multivariate extremal index of the Multivariate Maxima of Moving Maxima process.

Key Words: Multivariate extremal index, dependence conditions, multivariate extreme value theory.

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1 Introduction

Suppose \( X = \{X_n = (X_{n1}, \ldots, X_{nd})\}_{n \geq 1} \) is a \( d \)-dimensional stationary sequence with common distribution function \( F(x) = F(x_1, \ldots, x_d) \), \( x \in \mathbb{R}^d \). Let \( M_n = (M_{n1}, \ldots, M_{nd}) \) denote the vector of maxima where \( M_{nj} = \max\{X_{ij}, \ 1 \leq i \leq n\} \), \( j = 1, \ldots, d \). Denoting by \( \{X_n\}_{n \geq 1} \) the associated sequence of i.i.d. random vectors having the same \( d \)-dimensional d.f. \( F \), let \( \hat{M}_n = (\hat{M}_{n1}, \ldots, \hat{M}_{nd}) \) be the corresponding vector of maxima.

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If there exist sequences \( a = \{a_n = (a_{n1} > 0, \ldots, a_{nd} > 0)\}_{n \geq 1} \) and \( b = \{b_n = (b_{n1}, \ldots, b_{nd})\}_{n \geq 1} \) such that for \( u(x) = \{u_n(x) = (a_{n1}x_1 + b_{n1}, \ldots, a_{nd}x_d + b_{nd})\}_{n \geq 1} \),

\[
P(\hat{M}_n \leq u_n(x)) = P \left( \bigcap_{j=1}^{d} \{\hat{M}_{nj} \leq a_{nj}x_j + b_{nj}\} \right) \xrightarrow{n \to \infty} \hat{H}(x), \ x \in \mathbb{R}^d,
\]

where \( \hat{H} \) is a d.f. with non-degenerate margins, then the d.f. \( \hat{H} \) is a multivariate extreme value (MEV) distribution function, and one says that \( F \) is in the (multivariate) domain of attraction of \( \hat{H} \) (for the maxima). In particular, the univariate margins (or components) \( \hat{H}_j, j = 1, \ldots, d, \) of \( \hat{H} \) are extreme value distributions (Galambos, 1978; Resnick, 1987).

A MEV distribution function \( H \) can be characterized by its copula function \( D_H \) (or dependence function) which exhibits a number of interesting properties (Deheuvels, 1978; Hsing, 1989), namely its stability equation

\[
D_H^t(y_1, \ldots, y_d) = D_H(y_1^t, \ldots, y_d^t), \ \forall t > 0 \text{ and } (y_1, \ldots, y_d) \in [0, 1]^d. \quad (1.1)
\]

If the stationary sequence \( X \) satisfies some long range dependence conditions \( (D(u(x))) \) of Hsing (1989) or \( \Delta(u(x)) \) of Nandagopalan (1990) and

\[
P(M_n \leq u_n(x)) \xrightarrow{n \to \infty} H(x), \ x \in \mathbb{R}^d, \quad (1.2)
\]

where \( H \) is a d.f. with non-degenerate components, then the d.f. \( H \) is also a MEV distribution function.

The relation between the MEV distribution functions \( H \) and \( \hat{H} \) can be expressed through the function multivariate extremal index \( \theta(\tau) = \theta(\tau_1, \ldots, \tau_d), \ \tau \in \mathbb{R}_+^d \), introduced by Nandagopalan (1990).

A \( d \)-dimensional stationary sequence \( X \) is said to have a multivariate extremal index \( \theta(\tau) \in [0, 1] \) if \( \forall \tau = (\tau_1, \ldots, \tau_d) \in \mathbb{R}_+^d \), \( \exists u^{(\tau)}_n = (u^{(\tau_1)}_{n1}, \ldots, u^{(\tau_d)}_{nd}), n \geq 1, \) satisfying

\[
nP(X_{ij} > u^{(\tau_j)}_{nj}) \xrightarrow{n \to \infty} \tau_j, \ j = 1, \ldots, d, \quad (1.3)
\]

\[
P(\hat{M}_n \leq u^{(\tau)}_n) \xrightarrow{n \to \infty} \hat{G}(\tau) \quad (1.4)
\]