Linear programming, the simplex algorithm and simple polytopes

Gil Kalai

Institute of Mathematics, Hebrew University of Jerusalem, Jerusalem, Israel

Received 27 April 1997; accepted 7 May 1997

Abstract

In the first part of the paper we survey some far-reaching applications of the basic facts of linear programming to the combinatorial theory of simple polytopes. In the second part we discuss some recent developments concerning the simplex algorithm. We describe subexponential randomized pivot rules and upper bounds on the diameter of graphs of polytopes. © 1997 The Mathematical Programming Society, Inc. Published by Elsevier Science B.V.

Keywords: Simplex algorithm; Randomized pivot rule complexity; Combinatorial theory of simple polytopes

1. Introduction

A convex polyhedron is the intersection $P$ of a finite number of closed halfspaces in $\mathbb{R}^d$. $P$ is a $d$-dimensional polyhedron (briefly, a $d$-polyhedron) if the points in $P$ affinely span $\mathbb{R}^d$. A convex $d$-dimensional polytope (briefly, a $d$-polytope) is a bounded convex $d$-polyhedron. Alternatively, a convex $d$-polytope is the convex hull of a finite set of points which affinely span $\mathbb{R}^d$.

A (nontrivial) face $F$ of a $d$-polyhedron $P$ is the intersection of $P$ with a supporting hyperplane. $F$ itself is a polyhedron of some lower dimension. If the dimension of $F$ is $k$ we call $F$ a $k$-face of $P$. The empty set and $P$ itself are regarded as trivial faces. 0-faces of $P$ are called vertices, 1-faces are called edges and $(d-1)$-faces are called facets. For material on convex polytopes and for many references see Ziegler’s recent book [32].

The set of vertices and (bounded) edges of $P$ can be regarded as an abstract graph called the graph of $P$ and denoted by $G(P)$.

1 Email: kalai@math.huji.ac.il.
We will denote by \( f_k(P) \) the number of \( k \)-faces of \( P \). The vector \( (f_0(P), f_1(P), \ldots, f_d(P)) \) is called the \( f \)-vector of \( P \). Euler’s famous formula \( V - E + F = 2 \) gives a connection between the numbers \( V, E, F \) of vertices, edges and 2-faces of every 3-polytope.

A convex \( d \)-polytope (or polyhedron) is called simple if every vertex of \( P \) belongs to precisely \( d \) edges. Simple polyhedra correspond to non-degenerate linear programming problems. When you cut a simple polytope \( P \) near a vertex \( v \) with a hyperplane \( H \) which intersect the interior of \( P \), the intersection \( P \cap H \) is a \((d - 1)\)-dimensional simplex \( S \). The vertices of \( S \) are the intersections of edges of \( P \) which contain \( v \) with \( H \) and the \((k - 1)\)-dimensional faces of \( S \) are the intersection of \( k \)-faces of \( P \) with \( H \). The following basic property of simple polytopes follows:

- Let \( P \) be a simple \( d \)-polytope and let \( v \) be a vertex of \( P \). Every set of \( k \) edges adjacent to \( v \) determines a \( k \)-dimensional face of \( P \) which contains the vertex \( v \).

  In particular there are precisely \( \binom{d}{k} \) \( k \)-faces in \( P \) containing \( v \) and altogether \( 2^d \) faces (of all dimensions) which contain \( v \).

Linear programming and the simplex algorithm

Linear programming is the problem of maximizing a linear objective function \( \phi \) subject to a finite set of linear inequalities. The relevance of convex polyhedra to linear programming is clear. The set \( P \) of feasible solutions for a linear programming problem is a polyhedron.

There are two fundamental facts concerning linear programming the reader should keep in mind:

- If \( \phi \) is bounded from above on \( P \) then the maximum of \( \phi \) on \( P \) is attained at a face of \( P \), in particular there is a vertex \( v \) for which the maximum is attained. If \( \phi \) is not bounded from above on \( P \) then there is an edge of \( P \) on which \( \phi \) is not bounded from above.

- A sufficient condition for \( v \) to be a vertex of \( P \) on which \( \phi \) is maximal is that \( v \) is a local maximum, namely \( \phi(v) \) is bigger or equal than \( \phi(w) \) for every vertex \( w \) which is a neighbor of \( v \).

The simplex algorithm is a method to solve a linear programming problem by repeatedly moving from one vertex \( v \) to an adjacent vertex \( w \) of the feasible polyhedron so that in each step the value of the objective function is increased. The specific way to choose \( w \) given \( v \) is called the pivot rule.

The \( d \)-dimensional simplex and the \( d \)-dimensional cube

The \( d \)-dimensional simplex \( S_d \) is the convex hull of \( d + 1 \) affinely independent points in \( R^d \). The faces of \( S_d \) are themselves simplices. In fact, the convex hull of every subset of vertices of a simplex is a face and therefore \( f_k(S_d) = \binom{d+1}{k+1} \). The graph of \( S_d \) is the complete graph on \( d + 1 \) vertices.