On Upper Limits to the Difference in Bias Between 
Two Ratio Estimates

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Summary: In this short paper upper limits to the difference in bias between two ratio estimates relative to the standard error of the difference are given. The estimates compared may have their basis in any type of probability sample. When the two ratio estimates are either independent or negatively correlated the limit under consideration always exists and its expression is simple. On the other hand when the estimates are positively correlated the limit exists, but is subject to the rare qualification that the standard errors of the two estimates are unequal.

1. Introduction

From the data of sample surveys ratio estimates of the type \( r = \frac{y}{x} \), where \( y \) and \( x \) are estimates based on the same set of units, are often computed. It is well known that such ratio estimates are biased. For this reason the comparison of two ratio estimates by way of a difference may not be meaningful if the true difference is overwhelmed by the inherent bias. However such comparisons may be in order if the upper limits for such bias can be determined and shown to be relatively small.

For the purpose of a general discussion on upper limits to the difference in bias between two ratio estimates, the form which the estimates \( y \) and \( x \) (which are random variables) assume will not be discussed, except to mention that in most surveys of finite populations they are usually linear estimates based on a stratified sample whose units are drawn at each stage or phase, with equal or unequal probabilities, and with or without replacement of units.

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A comparison of two ratios $r_1 = y_1/x_1$ and $r_2 = y_2/x_2$, relating to some characteristic of a finite population, relevant to the following situations, is sometimes made.

(i) $r_1$ and $r_2$ may be based on two surveys at different periods of time which may be based on the same sample or a sample in which some of the ultimate sampling units of the first survey are retained at the second survey.

(ii) $r_1$ and $r_2$ may be based on two different surveys at different periods of time. That is, the surveys are different in the sense that the samples are independently drawn on each occasion and may not necessarily be based on an identical frame or even have the same number of ultimate units.

In a given survey, $r_1$ and $r_2$ may each relate to

(iii) two different strata, or

(iv) they may relate to domains of study which may even cut across strata.

In the light of the foregoing account, the following remarks which have a bearing on the problem will be made. In situations (ii) and (iii), $r_1$ and $r_2$ will be statistically independent in the context of the probabilities used in the selection of the units; therefore, the covariance of $r_1$ and $r_2$ will be zero, so that the correlation between $r_1$ and $r_2$ will be zero. However, this state of affairs does not obtain for situations (i) and (iv), and $r_1$ and $r_2$ will always be correlated, the magnitude and the sign of the correlation depending on the nature of the characteristics under study.

2. Solution of Problem

To obtain upper limits to the difference in bias between two ratio estimates

$r_1 = y_1/x_1$ and $r_2 = y_2/x_2$, whose true values are $R_1 = E(y_1)/E(x_1)$ and $R_2 = E(y_2)/E(x_2)$ respectively, we first consider the identity connecting the expectation of the ratio of two random variables $x$ and $y$, to the true value $R = E(y)/E(x)$ and the bias, which is

$$E(r) = E(x) + E(y) - E(x - y)$$

given by Koop [2]. The second term on the right hand side of this fundamental identity symbolizes the bias and yields the covariance expression

$$E \left[ y \left( \frac{1}{x} - \frac{1}{E(x)} \right) \right] = - \frac{1}{E(x)} \text{ cov} \left( \frac{y}{x}, x \right),$$