On a fundamental variational lemma for extremal quasiconformal mappings

RICHARD FEHLMANN

1. Introduction

In [R2] E. Reich considers the following extremal problem in qc (quasiconformal) mappings. Given are a closed set \( \sigma \) on the boundary \( \partial D \) of the unit disk \( D = \{ w \mid |w| < 1 \} \) which contains at least four points and a measurable set \( E \) in \( D \) where \( D \setminus E \) has positive area-measure and where, if \( \sigma \) is an infinite set, \( \tilde{E} \) is assumed to be compact in \( \hat{D} \setminus \sigma \). Furthermore a quasisymmetric boundary mapping \( h : \partial D \to \partial D \) is given and a measurable non-negative function \( b(w) \) on \( E \) with \( \text{ess sup}_{w \in E} b(w) < 1 \) which is called the "dilatation bound function": \( Q(h, \sigma, E, b) \) then denotes the class of all qc mappings \( F : D \to D \) which satisfy

\[ F|_{\sigma} = h|_{\sigma} \quad \text{and} \quad |\kappa_F(w)| \leq b(w) \quad \text{a.e. in } E, \]

where \( \kappa_F = F \bar{\alpha}/Fw \) is the complex dilatation of \( F \). In this class a mapping \( F \) is called extremal if it minimizes the value

\[ \text{ess sup}_{w \in D \setminus E} |\kappa_F(w)| \]

and is called uniquely extremal if it is the only such mapping.

In the case when \( E \) is the empty set a necessary and sufficient condition for extremality is the Hamilton-condition as has been shown in [H] and [RS]. In [R2] E. Reich has given a generalization of this condition which is necessary and sufficient for extremality in \( Q(h, \sigma, E, b) \) and by which extremal mappings can be characterized. But in his work an additional requirement had to be posed on \( b(w) \), namely that it is bounded away from zero. Later F. Gardiner succeeded in proving the analogous condition in the case when \( \sigma \) is finite and \( b(w) = 0 \) in \( E \) [G2]. He used a result from Teichmüller theory which he had proved in [G1].

In this note we use Gardiner’s result to generalize a fundamental variational lemma which is needed in Reich’s treatment. In its generalized form it turns out
to be adequate for the general case. In section 3 we apply it to handle the case where \( \sigma \) is infinite and \( b(w) = 0 \) in \( E \). The proof then follows exactly the same pattern as the one in Reich's paper. In a forthcoming paper of K. Sakan [Sa] it then will be applied to arbitrary dilatation bound functions \( b(w) \).

In section 4 we give, based on Reich's treatment, alternative proofs of Gardiner's result in two special cases. Namely, if the area-measure of the boundary \( \partial E \) of the set \( E \) is zero, then this result follows immediately by approximation and if \( E \) is supposed to be a closed set, it can be proved similarly.

Finally, I want to add that the idea of setting variable dilatation bounds as a side-condition for extremal problems goes back to O. Teichmüller ([T], p. 15), and to my knowledge R. Kühnau has been the first one who attacked such problems successfully. In [K1] he solved a problem of this sort (Satz 1) which enables him in [K2] to give a complete solution of our extremal problem above in the case where \( \sigma \) consists of four points by an essentially different method. No requirements as \( b(w) \geq \varepsilon > 0 \) had to be made except for some regularity assumptions on \( E \) and \( b(w) \).

2. Notations and the variational lemma

For a qc mapping \( F \) we denote its complex dilatation by \( \kappa_F \), the dilatation of \( F \) at the point \( w \) by \( D_F(w) = \frac{(1 + |\kappa_F(w)|)}{(1 - |\kappa_F(w)|)} \) and its maximal dilatation by \( K[F] \). We put \( \sigma' = h(\sigma) \), \( E_0 = \{ w \in E \mid b(w) = 0 \} \) and for a fixed element \( F \in Q(h, \sigma, E, b) \) we introduce

\[
f = F^{-1}, \quad \kappa = \kappa_F, \quad k_F = \text{ess sup}_{w \in D \setminus E} |\kappa_F(w)|
\]

and

\[
\hat{k}(z) = \begin{cases} 
\kappa(z) & z \in D \setminus F(E) \\
\frac{\kappa(z)}{b(f(z))} & z \in F(E \setminus E_0) \\
0 & z \in F(E_0)
\end{cases}
\]

(2.1)

We note that \( \| \hat{k} \|_\infty = \text{ess sup}_{z \in D} |\hat{k}(z)| = k_F \). Then the Banach-space \( B_{\sigma'} = \{ \varphi \mid \varphi \text{ holomorphic in } D, \| \varphi \| < \infty, \varphi dz^2 \text{ real along } \partial D \setminus \sigma' \} \) over the field \( \mathbb{R} \) will be used, where

\[
\| \varphi \| = \iint_{D} |\varphi(z)| \, dx \, dy, \quad (z = x + iy)
\]