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Feynman path integrals on phase space and the metaplectic representation

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1 Introduction

According to the WKB method a solution of Schrödinger's equation has the form

$$f(t_1,x_1) = \int_{x_0\in\mathbb{R}^n} e^{i\,S(t_1,x_1,t_0,x_0)/\hbar} A(t_1,x_1,t_0,x_0) f(t_0,x_0)\,dx_0 \tag{1.1}$$

asymptotically as $\hbar \to 0$. The exponent S satisfies the Hamilton-Jacobi equation from classical mechanics. According to the principal of stationary phase the major conribution to such an integral occurs at the critical points of the exponent.

Feynman in [15] (see also [17]) argued similarly. He argued that integrals

$$f(t_1,x_1) = \int_{x(t_1)=x} e^{i\,I(x)/\hbar} f(x(t_0))\mathscr{D}x \tag{1.2}$$

over the space of all curves $x : [t_0,t_1] \to \mathbb{R}^n$ are fundamental. He was led to integrals of this type by physical considerations. He assigned a phase $e^{iI(x)/\hbar}$ to each classical path x and summed over all paths x. The exponent $I(x)$ is the action integral from classical mechanics:

$$I(x) = \int_{t_0}^{t_1} K(\dot{x}) - V(x)\,dt$$

where K is the kinetic energy and V is the potential energy. The Euler-Lagrange equations of I are Newton's equations of motion. Hence by the principal of stationary phase the major contribution to (1.2) should occur at the classical trajectories.

The generating function S of equation (1.1) is obtained from the action integral $I(x)$ of equation (1.2) by evaluating $I(x)$ at 'the' classical trajectory x

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such that $x(t_0) = x_0$ and $x(t_1) = x_1$. If Feynman's integral were finite dimensional one might integrate out all the variables except x_0 and arrive at (1.1). Something analogous occurs in Hörmander's theory [21] of Fourier integral operators where the phase function which appears in the expression for a Fourier integral operator can be replaced by another phase function which defines the same symplectic relation. In fact, this is almost exactly what Feynman did. He replaced the integral $I(x)$ by a finite dimensional approximation and evaluated the resulting finite dimensional integral to get something of form (1.1).

In this paper we do something similar to what Feynman did. Unlike Feynman, we use paths in phase space rather than configuration space and use the symplectic action integral rather than the (classical) Lagrangian integral. We eventually restrict to (inhomogeneous) quadratic Hamiltonians so that the finite dimensional approximation to the path integral is a Gaussian integral. In evaluating this Gaussian integral the signature of a quadratic form appears. This quadratic form is a discrete approximation to the second variation of the action integral. We obtain the formula of Leray [24] for the Metaplectic representation.

For Lagrangians of the form kinetic energy minus potential energy, evaluated on curves in configuaration space, the index of the second variation is well-defined and, via the Morse Index Theorem, related to the Maslov Index of the corresponding linear Hamiltonian system. The second variation of the symplectic action has both infinite index and infinite coindex. However, this second variation does have a well-defined signature via the aforementioned discrete approximation. This signature can be expressed in terms of the Maslov index of the corresponding linear Hamiltonian system. This is a symplectic analog of the Morse Index Theorem.

Our topic has a vast literature. The book of Simon [36] has a bibliography of 291 items. Much of this work (e.g. [3], [4], [5], [6], [7], [8], [16], [18], [28], [29]) is directed at defining the Feynman integral for as large a class of integrands as possible. Our formula for the metaplectic representation appears in [24] where it is obtained by other arguments. Souriau [38] found an explicit solution for the quantum harmonic oscillator involving the Maslov index (thus correcting Feynman's original formula which is valid only for short times). Keller [22] first noticed the phase shift due to the Maslov index in Theorem 8.5 below and for this reason the Maslov index is sometimes called the *Keller-Maslov index*. Duistermaat's article [14] explains how to interpret the Morse index in terms of the Maslov index but in the situation studied here the Morse index is undefined. The article [1] explains how Feynman and Dirac [11] were motivated by using the method of stationary phase to obtain classical mechanics as the limit (as $\hbar \to 0$) of quantum mechanics. Most work to date uses Hamiltonians of the form kinetic energy plus potential energy and is limited to integration over paths in configuration space, but Daubechies and Klauder [12] (see also [13]) have formulated a theory of path integrals on phase space where the Hamiltonian function can be any polynomial. They remark that the 'time slicing' construction used by Feynman does not generalize. However, our Hamiltonians are at worst quadratic and Feynman's original method is adequate.