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Rational evaluation subgroups

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1 Introduction

In this paper, we study Gottlieb's evaluation subgroups of homotopy groups [6] in the context of rational homotopy theory. Following Federer's approach to the study of function spaces [2], we derive easily calculable conditions on the rational homotopy of a map $f : X \to Y$ which imply the vanishing or nonvanishing of the corresponding evaluation subgroup of Y. Our results establish a precise relationship between the "freeness" of the image of the rational homotopy of X under f in Y and the vanishing of the corresponding rational evaluation subgroup of Y. As applications, we extend the examples of spaces with vanishing Gottlieb groups given in [3, §7] to other evaluation subgroups, make explicit calculations of rational homotopy groups of function spaces, and show how, for certain classes of spaces, the rational Gottlieb groups faithfully detect the important division in rational homotopy theory between elliptic and hyperbolic spaces.

Given connected CW complexes X and Y, let $M(X,Y)$ denote the space of continuous functions from X to Y with the compact-open topology. Given $f : X \to Y$, write $M_f(X,Y)$ for the path component of f in $M(X,Y)$. For each $f : X \to Y$, evaluation at the basepoint of X defines a fibration $\rho : M_f(X,Y) \to Y$ called the *evaluation fibration.* We refer to the subgroups $\rho_*(\pi_*(M_f(X,Y)))$ of $\pi_*(Y)$ as the *evaluation subgroups* and the subgroups $\rho_*(\pi_*(M_f(X,Y)) \otimes \mathbb{Q})$ of $\pi_*(Y) \otimes \mathbb{Q}$ as the *rational evaluation subgroups* of Y. In the special case $X = Y$ and $f = 1_X$, the evaluation subgroup $\rho_*(\pi_*(M_1(X,X)))$ of $\pi_*(X)$ is called the *Gottlieb group* and written $G_*(X)$.

The Gottlieb group $G_*(X)$ satisfies a universal property with respect to fibrations that derives from the presence of the space of self-equivalences of X as fibre in the universal fibration for X [5]. Rationally, this universal property implies that the vanishing of the $G_*(X) \otimes \mathbb{Q}$ is equivalent to the splitting of the rational homotopy exact sequence for any fibration with fibre X. Thus the Gottlieb group is a natural object for study in rational homotopy theory. In [3], Felix

and Halperin proved the remarkable result that, for spaces X of finite rational category n, $G_*(X)\otimes\mathbb{Q}$ is concentrated in odd degrees and of rank $\leq n$. Their result is a consequence of their mapping theorem for rational category and requires essential use of the universal property of the Gottlieb group. While evaluation subgroups other than the Gottlieb group have not been studied rationally, evaluation fibrations without restriction on components have been a central object for the study of homotopy properties of function spaces. For example, Federer constructed his spectral sequence for function spaces [2] by factoring the evaluation fibration through a series of intermediary fibrations. Evaluation fibrations also appear as the focal point of Hansen's classification theorems in [8, 9, 10]. Our purpose here is to apply function space techniques – specifically, Federer's techniques – to the study of rational evaluation subgroups to obtain results which are not limited to the Gottlieb group. As we shall see, our approach provides a new ease of calculation when applied to the rational Gottlieb group, as well.

The paper is organized as follows. In §2 we prove that, for suitably restricted spaces X and Y, the rational evaluation subgroups are invariants of the rational homotopy of maps $f : X \to Y$. Thus, in particular, the vanishing or nonvanishing of the rational evaluation subgroup corresponding to $f : X \to Y$ is captured by the Quillen model of $f : X \to Y$. As a direct consequence of Federer's approach, which we recall in §3, we prove (Theorem 4.1) that if Y has finite-dimensional rational homotopy then its rational evaluation subgroups are always nontrivial. Using Neisendorfer's refinement of Quillen's theory [14] combined with Federer's approach, we prove a partial converse to this result; namely (Theorem 4.2) we determine a simple condition on the Quillen model of $f : X \to Y$ which implies that the corresponding rational evaluation subgroup vanishes. In §5, we apply Theorem 4.2 to extend certain known examples of vanishing rational Gottlieb groups to other evaluation subgroups and to calculate the rational homotopy groups of function spaces for a large class of rational co-H-spaces. Our results in §4 also apply to give new examples of how the rational homotopy of the space of self-maps of a complex X can be seen to characterize the rational homotopy of X. In [20], we showed how the rational homotopy of the space M(X,X) for suitable X faithfully detects the presence of a rational H-space structure on X. Applying Theorems 4.1 and 4.2, in §5 we prove that for the classes of spaces studied by Neisendorfer in [15] the vanishing of $G_*(X)\otimes\mathbb{Q}$ is equivalent to X being rationally hyperbolic. (Recall from [4], a finite simply connected complex X is *rationally elliptic* if $\dim\pi_*(X)\otimes\mathbb{Q} < \infty$ otherwise X is *rationally hyperbolic.*) Thus, for these classes of spaces, not only is the vanishing of the rational Gottlieb group a consequence of conditions on the Quillen model of X, it in fact implies these conditions. We conclude with examples to show that this situation does not obtain for the class of spaces consisting of all finite simply connected complexes and that the conditions of Theorem 4.2 are not necessary conditions for the vanishing of the rational Gottlieb group.