Embedding Problem and Functional Equations

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Abstract. In this paper we give a method for solving the functional equations arising from the differential embedding problem. We also obtain the conditions for embedding one-dimensional diffeomorphisms into differential flows.

§1. Introduction

In this paper, we consider how to embed diffeomorphisms into differential flows.

For the continuous embedding problem, there are many results for 1-dimensional homeomorphisms ([1]–[3]). Some authors even consider how to embed continuous maps into semi-flows (see [2], [4]). However, most of these works are based on the idea of Sternberg [5].

For the differential embedding problem, Palis [6] pointed out that the diffeomorphisms which can be embedded into flows generated by vector fields with some smoothness are "few". In [7], Lam considered how to embed $C^1$ diffeomorphisms on intervals of $R^1$ into flows with some smoothness. He also obtained the conditions for embedding $C^1$ diffeomorphisms on intervals into $C^1$ flows ([8],[9]).

As the differential embedding problem is meaningful in the study of non-local bifurcation theory, we will use a basic relation between flows and vector fields to consider this problem. In Sect. 2 we show that a function on $C$ actually solves the embedding problem. This answers a question posed by the authors of [10]. In Sect.3, we obtain the solutions of 1-dimensional embedding equations. Furthermore, we can use these solutions to obtain the conditions for embedding $C^2$ diffeomorphisms on intervals into $C^1$ vector fields (see Theorem 2). Sect.4 contains some discussions on the conditions in Sect.2 and some extensions of Theorem 2.

§2. Embedding Equations

Let $M$ be a smooth manifold. Let $f \in Diff^r(M)$ be a $C^r$ diffeomorphism. We say that $f$ can be embedded into $C^r$ flows if there is a $C^r$ flow $F : R \times M \rightarrow M$ such that
$F_t(\cdot) = F(\cdot, 1) = f$. Of course, the case $r = 0$ means that $f$ is a homeomorphism and $F$ a continuous flow. In this paper, we say that $f \in \text{Diff}^r(M)(r \geq 1)$ is embedded into a $C^*$ vector fields $V \in X^s(M)(0 \leq s < r)$ if $f$ is embedded into the flow $\{F_t\}$ generated by $V$, i.e., $\{F_t\}$ satisfies
\begin{equation}
\frac{dF_t(x)}{dt} = V(F_t(x)), \quad t \in \mathbb{R}^1, \quad x \in M
\end{equation}
and
\begin{equation}
F_0 = \text{id}.
\end{equation}

It is well known that if $\{F_t\}$ is the flow generated by $V \in X(M)$, then $V$ must satisfy the following functional equation:
\[ V(F_t(x)) = DF_t(x)V(x), \quad t \in \mathbb{R}^1, \quad x \in M. \]

Especially, if $f \in \text{Diff}^r(M)$ is embedded into a vector field $V \in X(M)$, then $V$ satisfies the following embedding equation:
\begin{equation}
V(f(x)) = Df(x)V(x), \quad x \in M.
\end{equation}

Of course, for a given $f \in \text{Diff}^r(M)$, in general we don't know whether a solution $V$ of Eq.(3) solves the embedding problem for $f$. However, the function $\phi$ defined in Sect. 2 in [10] actually solves the embedding problem.

Let us first formulate the question. Let $w(z)$ be an analytic function near $z = 0$ and $w^r(z)$ its $r$th iteration ($r = 0, 1, 2, \ldots$). If $w(0) = 0$ and $w'(0) = \alpha$ for some real number $\alpha \in (0, 1)$, McKiernan (1963) proved that the series
\begin{equation}
\phi(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sum_{i=0}^{n} \binom{n}{i}(-1)^{n-i}w^i(z)
\end{equation}
converges in some neighborhood $B_0$ of $z = 0$ and $\phi$ satisfies
\begin{equation}
\phi(w(z)) = w'(z)\phi(z), \quad z \in B_0.
\end{equation}

We can show that $\phi$ solves the embedding problem for $w$, i.e., $\phi$ generates a local flow $w_t(z)$ such that $w_1 = w$.

Firstly we have
\begin{equation}
\phi'(0) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sum_{i=0}^{n} \binom{n}{i}(-1)^{n-i} \alpha^i = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(-1 + \alpha)^n,
\end{equation}
i.e.
\begin{equation}
\phi'(0) = \ln \alpha.
\end{equation}

Since 0 is an attracting fixed point of $w$, by the complex dynamics (Theorem 3.3 in [11]), we know that
\begin{equation}
h_n(z) = \frac{w^n(z)}{\alpha^n}
\end{equation}