On the Pontrjagin Classes of a Submanifold *)

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Abstract. In this paper the Pontrjagin forms of a closed oriented manifold immersed in $R^n$ are expressed in terms of Plücker coordinates.

1. Grassmannians in Plücker Coordinates

Let $R^n$ be a real euclidean vector $n$-space with an orthonormal basis $(\delta_1, \ldots, \delta_n)$. So

$$\delta_{\mu_1} \wedge \cdots \wedge \delta_{\mu_m}, \quad 1 \leq \mu_1 < \cdots < \mu_m \leq n$$

is an orthonormal basis of $\wedge^m R^n$.

Convention. All manifolds and maps are $C^\infty$; the euclidean scalar product is denoted by a dot; indices have the ranges

$$1 \leq \mu, \nu, \ldots \leq n; \quad 1 \leq i, j, \ldots \leq m < \alpha, \beta, \ldots \leq n;$$
repetition of indices implies summation. Thus

$$(\delta_{\mu_1} \wedge \cdots \wedge \delta_{\mu_m}) \cdot (\delta_{\nu_1} \wedge \cdots \wedge \delta_{\nu_m}) = \delta_{\mu_1 \cdots \mu_m},$$

$$(\delta_{\mu_1} \wedge \cdots \wedge \delta_{\mu_m}) \cdot (\delta_{\nu_1} \wedge \cdots \wedge \delta_{\nu_m}) = \frac{n!}{(n-m)!}.$$

Let

$$R^n_m = SO(n) / SO(m) \times SO(n-m)$$

be the Grassmann manifold of oriented subspaces of dim $m$ in $R^n$. Every point $p \in R^n_m$ has an orthocomplement $p^\perp \in R^n_{n-m}$, and

$$p \mapsto p^\perp : R^n_m \rightarrow R^n_{n-m}$$

is a diffeomorphism, which identifies the two manifolds.

Let $(e_i) = (e_1, \ldots, e_n)$ be orthonormal frames in $R^n$ adapted to $R^n_m$, i.e., $(e_i)$ is a basis of a subspace $p \in R^n_m$ and so $(e_\alpha)$ is a basis of $p^\perp \in R^n_{n-m}$. Hence a point $p \in R^n_m$ is uniquely represented by a unit $m$-vector

$$p = e_i \wedge \cdots \wedge e_m \in \wedge^m R^n.$$

Let

$$e_\mu = \delta_\nu a_{\nu \mu}.$$ 

Then

$$a_{\nu \mu} = \delta_\nu \cdot e_\mu, \quad a_{\nu \mu} a_{\nu \nu} = \delta_{\mu \mu}.$$
Let
\[ de_\mu = e_\mu \omega_\mu. \]

Then
\[ d\omega_\mu + \omega_\mu \wedge \omega_\mu = 0, \quad \omega_\mu = e_\mu \cdot de_\mu = -\omega_\mu. \]

A tangent vector \( dp \) of \( R^n_m \) at the point \( p \in R^n_m \) is represented as an \( m \)-vector
\[ dp = d(e_1 \wedge \cdots \wedge e_m) = e_1 \wedge \cdots \wedge e_m \wedge e_\mu \omega_\mu. \]

Hence the \( m \)-vectors
\[ E_\alpha = e_1 \wedge \cdots \wedge e_\alpha \wedge \cdots \wedge e_m \]
constitute a frame in \( TR^n_m \), the tangent bundle of \( R^n_m \), and its coframe consists of the forms
\[ \omega_\alpha = E_\alpha \cdot dp = (e_1 \wedge \cdots \wedge e_\mu \wedge \cdots \wedge e_m) \cdot dp. \]

By definition, the Plücker coordinates of a point \( p \in R^n_m \) relative to the basis \( (\delta_\mu) \) of \( R^n \) are the components of the alternating tensor
\[ p_{\mu_1 \cdots \mu_m} = (\delta_{\mu_1} \wedge \cdots \wedge \delta_{\mu_m}) \cdot (e_1 \wedge \cdots \wedge e_m) = \det (\delta_{\mu_i} \cdot e_j) = \det (a_{\mu i}). \]

The coordinates with respect to the moving frame \( (e_\mu) \) will be indicated by the mark \( (e) \). Thus
\[ p_{\mu_1 \cdots \mu_m}^{(e)} = \delta_{\mu_1 \cdots \mu_m}^{1 \cdots m}, \]
\[ p \cdot p = \sum_{\mu_1 < \cdots < \mu_m} p_{\mu_1 \cdots \mu_m} p_{\mu_1 \cdots \mu_m} = \sum_{\mu_1 < \cdots < \mu_m} p_{\mu_1 \cdots \mu_m} p_{\mu_1 \cdots \mu_m} = 1. \]

The Plücker coordinates \( p_{\mu_1 \cdots \mu_m} \) satisfy the Plücker relations
\[ p_{\mu_1 \cdots \mu_{m-1} \mu_m \mu_{m-1}} = 0, \]
where the brackets mean an alternation of the indices. So, \( p_{\mu_1 \cdots \mu_m} p_{\mu_1 \cdots \mu_m} \) is a curvature-type tensor which satisfies the first Bianchi identity.

In the natural frame \( (\delta_\mu) \),
\[ d(p_{\mu_1 \cdots \mu_m}) = (\delta_{\mu_1} \wedge \cdots \wedge \delta_{\mu_m}) \cdot dp = (dp)_{\mu_1 \cdots \mu_m}, \]
it is convenient to write this simply as \( dp_{\mu_1 \cdots \mu_m} \). So
\[ dp_{\mu_1 \cdots \mu_m} = (\delta_{\mu_1} \wedge \cdots \wedge \delta_{\mu_m}) \cdot (e_1 \wedge \cdots \wedge e_\mu \wedge \cdots \wedge e_m) \omega_\alpha = (E_\alpha)_{\mu_1 \cdots \mu_m} \omega_\alpha; \]

While in the moving frame \( (e_\mu) \),
\[ d(p_{\mu_1 \cdots \mu_m}^{(e)}) = d(\delta_{\mu_1 \cdots \mu_m}^{1 \cdots m}) = 0, \]
\[ (dp)_{\mu_1 \cdots \mu_m}^{(e)} = (e_1 \wedge \cdots \wedge e_\mu) \cdot (e_1 \wedge \cdots \wedge e_\alpha \wedge \cdots \wedge e_m) \omega_\alpha = (E_\alpha)_{\mu_1 \cdots \mu_m} \omega_\alpha, \]
\[ (dp)_{\mu_1 \cdots \mu_m}^{(e)} = \omega_\alpha, \]
i.e.,
\[ \omega_\alpha = (-1)^{\mu-1} (dp)^{(e)}_{\mu_1 \cdots \mu_m}. \]

Note that he grassmannian \( R^n_m \) represented in the Plücker coordinates \( p_{\mu_1 \cdots \mu_m} \) has the induced metric