PERTURBATION THEORY FOR PARTICLES IN ONE-DIMENSIONAL, CENTRAL- OR AXIAL-SYMMETRIC FIELDS

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A perturbation theory for which all corrections to a nonperturbed state are defined by its wave function is obtained and a new applicability condition for this theory is formulated. By considering a simple example, a comparison with the Rayleigh–Schrödinger perturbation theory, giving an analytical expression for the corresponding series, is performed.

According to the Rayleigh–Schrödinger perturbation theory, one needs to know the wave functions and the energies of all other nonperturbed states in order to calculate the corrections to a state \( |n⟩ \) with an energy \( E_n \). In the cases of one-dimensional, central- or axial-symmetric fields, to be considered below, the Schrödinger equation is reduced to an equation of the form

\[
\mathcal{L} y = y'' + \varphi(x)y = 0,
\]

and the perturbative corrections are defined by the equation

\[
\mathcal{L} = f(x, \alpha).
\]

The corresponding corrections to the energy enter \( f \) as parameters \( \alpha \). As is known, all solutions to inhomogeneous equation (1') are defined by only one solution to homogeneous equation (1). Therefore, all corrections have to be expressed solely in terms of the wave function of the nonperturbed state. This idea was developed in [1] for a central-symmetric field.

We use a different approach, which includes one-dimensional and axial-symmetric fields as well. The nonperturbed function obeys the following boundary conditions:

\[
y_0(a) = y_0(\infty) = 0,
\]

where \( a = -\infty \) for the one-dimensional field and \( a = 0 \) for the two other cases.

1. One special solution \( y^* \), which satisfies the initial conditions \( y(\xi) = y'(\xi) = 0 \), is given by the Duhamel principle,

\[
y^*(x, \xi) = \int_{\xi}^{x} u(x, x') f(x', \alpha) dx', \quad u(x, x') = y_{02}(x)y_{01}(x') - y_{01}(x)y_{02}(x').
\]

Here \( y_{01} = y_0 \) and \( y_{02} \) are two linearly independent solutions to (1) with the unit Wronskian that yields

\[
y_{02}(x) = y_0(x) \int_{\xi}^{x} \frac{dx''}{y_0^2(x''')} \forall x.
\]

Taking (3) into account, one can rewrite the solution to (2) in the following form:

\[
y^*(x, \xi) = y_0(x) \int_{\xi}^{x} \frac{dx''}{y_0^2(x''')} \int_{\xi}^{x''} y_0(x') f(x', \alpha) dx'.
\]

With regard to (2), a general solution to (1') can be written as

\[
y(x, p, q) = -y_{01}(x) \int_{p}^{x} dx' y_{02}(x') f(x', \alpha) + y_{02}(x) \int_{q}^{x} dx' y_{01}(x') f(x', \alpha),
\]

where \( p \) and \( q \) are arbitrary constants with \( y^*(x, \xi) = y(x, \xi, \xi) \).

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2. Now consider the boundary problem

$$\tilde{C} y = f(x, \alpha), \quad y(\alpha) = y(\infty) = 0,$$

(4)

and assume that $y_0$ satisfies both boundary conditions, while $y_{02}$ satisfies none of them. This assumption defines the constant $q$, which is equal to $a$, and the parameter $\alpha$ in (3'),

$$\int_a^\infty dx' y_0(x') f(x', \alpha) = 0. \quad (5)$$

The solution to (4) has the form

$$y(x, p, a) = -y_{01}(x) \int_p^x dx' y_{02}(x') f(x', \alpha) - y_{02}(x) \int_x^\infty dx' y_{01}(x') f(x', \alpha). \quad (6)$$

3. In the fields under consideration, the Schrödinger equation is reduced to a one-dimensional equation. Let the field be of the form $V = V_0 + \varepsilon v$, where $\varepsilon v$ is a small perturbation. For a fixed state, whose index $n$ is omitted for brevity, we have $E = E_0 + \varepsilon E^{(1)} + \varepsilon^2 E^{(2)} + \cdots$, $y = y_0 + \varepsilon y^{(1)} + \varepsilon^2 y^{(2)} + \cdots$, where $E$ is the energy and $y, x^{1/2}y,$ and $x^{-1}y$ are the wave functions of the one-dimensional and radial equations respectively (in last two cases, $x$ is the radial coordinate). The equations for $E^{(k)}$ and $y^{(k)}$ have the form (1'),

$$f_1 = 2\Delta v y_0, \quad f_2 = 2\Delta v y^{(1)} - 2E^{(2)} y_0, \quad \ldots, \quad \Delta v = v(x) - E^{(1)} = v(x) - \langle y_0 v | y_0 \rangle. \quad (7)$$

The function $y_0$ is normalized, while $y$ is assumed to satisfy the condition $\langle y_0 | y \rangle = 1$ that yields

$$\langle y_0 | y^{(k)} \rangle = \int_a^\infty dx y_0(x) y^{(k)}(x) = 0, \quad k > 0, \quad (8)$$

$$E^{(k)} = \langle y_0 | y^{(k-1)} \rangle. \quad (9)$$

The Rayleigh–Schrödinger perturbation theory is based on the expansion of $y^{(k)}$ in terms of all functions $y_{0n'}$ of nonperturbed states $n'$. In our case, the solutions for $y^{(k)}$ and $E^{(k)}$ are given by Eqs. (6)–(9) and are expressed only in terms of $y_0$. According to (6),

$$y^{(1)}(x) = -2 \left[ y_0(x) \int_p^x dx' y_0(x') y_{02}(x') \Delta v(x') + y_{02}(x) \int_x^\infty dx' y_0^2(x') \Delta v(x') \right], \quad (10)$$

and the constant $p$ is defined by condition (8). In central fields, the first integral in (10) is meaningful for $p = a = 0$ if $\lim_{x \to 0} x^2 v(x) = 0$. Therefore, in accordance with (10), we have

$$y^{(1)}(x) = cy_0(x) + *y^{(1)}(x, 0) = y_0(x)[c + F_1(x)], \quad (10')$$

where

$$F_1(x) = \int_0^x \frac{F(x') dx'}{y_0^2(x')}, \quad F(x) = 2 \int_0^x y_0^2(x') \Delta v(x') dx'. \quad (11)$$

It follows from (5) that $F(\infty) = 0$. The constant $c$ is defined by condition (8) at $k = 1$, such that

$$y^{(1)}(x) = y_0(x) \Delta F_1(x), \quad \Delta F_1(x) = F_1(x) - F_1, \quad F_1 = \int_0^\infty y_0^2(x) F_1(x) dx. \quad (12)$$

According to (9), for $k = 2$, we have

$$E^{(2)} = \langle y_0 | y | y^{(1)} \rangle = \int_0^\infty y_0^2(x) v(x) (F_1(x) - F_1) dx = \int_0^\infty y_0^2(x) \Delta v(x) F_1(x) dx =$$

$$= \frac{1}{2} \int_0^\infty F'(x) F_1(x) dx = -\frac{1}{2} \int_0^\infty \frac{F^2(x) dx}{y_0^2(x)} < 0. \quad (13)$$