BOUND STATE SOLUTIONS OF THE ELLIPTIC SINE-GORDON EQUATION

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We present a detailed investigation of finite-energy solutions with point-like singularities of the elliptic sine-Gordon equation in a plane. Such solutions are of the bound-state type in the sense of scalar field theory. If the solution has a unique singularity, then it behaves as a soliton-like annular wave packet at a large distance from the singularity. The effective radius of this wave packet is evaluated both analytically and numerically for axially symmetric solutions. The analytical investigation is based on the method of isomonodromy deformations for the third Painlevé equation, which singles out these solutions as separatrices of the manifold of general solutions (with infinite energy). Exact analytical estimates provide a tool for investigating bound-state solutions of the nonintegrable sine-Gordon equation with a nonzero right-hand side. More precisely, for large-intensity fields at the singularity, we derive the critical forcing that allows the existence and stability of a bound state. As an illustration, we consider two applications: a large-area Josephson junction and a nematic liquid crystal in a rotating magnetic field. For each of the examples, we evaluate the critical values of the field that allow finite-energy regimes. These are in good agreement with numerical and experimental data.

1. Introduction

Singular solutions of the elliptic sine-Gordon equation

\[ \varphi_{xx} + \varphi_{yy} = \sin \varphi - f, \quad f = \text{const.} \]  

with singularities of the form

\[ \varphi = a \log \frac{1}{r} + \alpha + o(1), \quad r^2 = x^2 + y^2, \quad r \to 0, \]  

are used for modeling point defects in condensed matter physics. Solutions of this type describe, for example, the inhomogeneity of the superconducting current near the corners of large rectangular Josephson junctions \([1, 3]\), as well as the behavior of the director in a nematic liquid crystal in a rotating magnetic field near the impurities \([4, 7]\). In addition, Eq. (1.1) can be derived from a certain approximation \([6, 8]\) to the anisotropic Landau-Ginzburg equation, where it provides a simple model for phase defects.

Another problem in which singular solutions are relevant describes bound states in scalar field theory. In the class of smooth solutions, Eq. (1.1) is known to have no bound states \([9, 10]\) in \(\mathbb{R}^2\), i.e., no solutions with the finite energy

\[ E = \int \left[ \frac{1}{2} \varphi_x^2 + \frac{1}{2} \varphi_y^2 - \cos \varphi - \cos \varphi_0 - f(\varphi - \varphi_0) \right] dx \, dy \]  

providing a minimum of (1.3), that is, \(\delta^2 E / \delta \varphi^2 \geq 0\). At infinity, the field \(\varphi\) reaches an asymptotically homogeneous state: \(\varphi \to \varphi_0, \quad r \to \infty\). However, once solutions with point-like singularities (1.2) are
allowed, global finite-energy solutions exist, which means that bound states become possible. Then we have to eliminate a small disk or radius \( a \ll 1 \) around the singularity from the integration domain in (1.3), where \( a \) is the “cutoff” radius determined by the physical applicability of the continuous model (1.1) [11].

The general structure of singular solutions was determined numerically in [2, 3, 7] for Eq. (1.1) and its \((2+1)\)-dimensional version.

\[
-\varphi_t + \varphi_{xx} + \varphi_{yy} = \sin \varphi - f.
\]  

(1.4)

whose stationary states are described by (1.1). For a free field (i.e., for \( f = 0 \)), such solutions exist in the radially symmetric case and are bounded in \( \mathbb{R}^2 \setminus \{x^2 + y^2 < \varrho^2\} \), decreasing exponentially at infinity, \( x^2 + y^2 \to \infty \). As long as the distance from the center is not too large, an intermediate estimate of the type of applicable radially symmetric wave packet can be interpreted as a system of annular solitons of Eq. (1.1) with a common center at the singularity, i.e., as a “target-like” structure [2-5]. The driving force \( f \) plays a crucial role regarding the existence of finite-energy solutions to (1.1). It was found in [3, 7] that for a value of \( f \) above a certain critical value \( f_0 \), the stationary solutions become unstable and give rise to new dynamic regimes in Eq. (1.4).

In what follows, we give a detailed analysis of singular solutions for the radially symmetric equation (1.1).

\[
\varphi_{rr} - \frac{1}{r} \varphi_r = \sin \varphi - f.
\]  

(1.5)

For the value \( f = 0 \), Eq. (1.5) reduces to a special case of the Painlevé-III equation (PIII). Its general solutions have been the subject of analytical investigations in the framework of the isomonodromy deformation method [12-14]. This technique is applicable, in particular, to bound-state solutions (see Sec. 2.2 below) that play the role of a one-dimensional “separatrix” manifold in the two-dimensional manifold of general solutions to (1.5). Their behavior at infinity, where they fall off asymptotically as

\[
\varphi(r) = 2\pi k - \gamma \sqrt{\frac{\pi}{2r}} e^{-r} [1 - O(r^{-1})], \quad r \to \infty.
\]  

(1.6)

(with \( k \) being an integer), gives rise to an equation for two first-integrals (the monodromy data). This equation, in turn, allows one to explicitly construct the relation between the \( \alpha \) and \( \beta \) constants from the asymptotic formula (1.2) at the singular point. Another consequence of this parameterization by the monodromy data are the formulas that relate \( k \) and \( \gamma \) from the asymptotic formula (1.6) to the initial data \( \alpha \) and \( \beta \).

Further, the explicit relations between the parameters of (1.2) and (1.6) allow us to evaluate the effective radius of the bound state, which, roughly speaking, is the coordinate of the remotest annular soliton. This provides a tool for investigating bound-state solutions in the integrable equation (1.5) with an external force \( f \). Namely, for large \( \alpha \) and \( f \ll 1 \), we obtain the critical value \( f_0(\alpha) \approx \text{const} \cdot \alpha^{-1} \) that controls the existence and stability of solutions (1.6) (see Sec. 3). In Sec. 4, this effect is illustrated in the example of two concrete physical applications: large-area Josephson junctions and nematic liquid crystals in a rotating magnetic field. In each of these examples, we evaluate the critical values of the field and of the external force that allow finite-energy regimes to exist. They demonstrate good agreement with computer simulations and with experimental data [3, 7].

To conclude this section, let us point out the relation between the solutions that we consider here and the bound state solutions of the sinh-Gordon equation (which is obtained from Eq. (1.5) by substituting the hyperbolic sine for the sine on the right-hand side). Those solutions to the sinh-Gordon equation that decrease exponentially at infinity were studied in detail in [15]. Under the mapping \( \varphi \to i \varphi \), they correspond to infinite-energy solutions of (1.5). Note, however, that the relation for the amplitude \( \gamma = \gamma(\alpha, \varphi) \) from (1.6) is correctly reproduced by this mapping (see [13]).