SPECTRAL PROPERTIES OF HAMILTONIANS WITH A MAGNETIC FIELD AT A FIXED PSEUDOMOMENTUM. I

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It is shown that the energy operator of an n-particle neutral system with a fixed pseudomomentum in a homogeneous magnetic field can be written as an operator in the space of the relative motion. The Hunziker–Van Winter–Zhislin theorem on localization of the essential spectrum is proved for this operator with regard to the permutation symmetry for all $n \geq 2$. Conditions for the finiteness and infiniteness of the discrete spectrum and spectral asymptotic formulas with estimates for the remainder are established for the case $n = 2$. In particular, these results can be applied to the Hamiltonian of the hydrogen atom in a homogeneous magnetic field.

Introduction

The main difficulty in studying the spectra of multiparticle systems in magnetic fields lies in the impossibility of separating the motion of the center-of-mass [1] in the general case. As a result, the Hunziker–Van Winter–Zhislin theorem on localization of the essential spectrum proves to be false for the situation in question. In turn, the fact that the essential spectrum is unknown does not allow one to study the structure of the discrete spectrum either. What has been said relates to systems in homogeneous magnetic fields where not all particles are identical and to arbitrary systems in weakly decaying magnetic fields. (However, everything proceeds well in the case of a field increasing at infinity on the $xy$-plane [2, 3]).

There are two methods for overcoming this difficulty. The first was suggested in [1] for a homogeneous magnetic field. This method consists of restricting the operators under consideration to the subspace of functions with a fixed pseudomomentum. It permits systems with arbitrary charges (including neutral ones) to be considered. However, the obstacles emerging in its application are so serious that no practical results have yet been obtained regarding the discrete spectra of $n$-particle systems with a fixed pseudomomentum, even in the case $n = 2$.

The other method, which was put forward in [2–4], uses the $SO(2)$ symmetry of the system. It is based on studying the spectra of multiparticle Hamiltonians in subspaces of functions that are transformed according to fixed-weight representations of the group $SO(2)$. The essential spectrum can be found using this method, not only in the case of a homogeneous field, but, also, for slowly decaying fields. However, it does not apply to neutral systems (such as atoms or molecules).

It is for this reason that we resort to the method of a fixed pseudomomentum suggested in [1] since it provides the most universal approach for systems in a homogeneous magnetic field. However, in contrast to [1], where the methods of integral equations were used and a purely functional approach was employed, we systematically apply geometric methods. Only the general idea of restricting the operators in question to the space of functions with a fixed pseudomomentum is, in fact, taken from [1].

We confine ourselves to considering only the case of neutral systems because it is particularly important for applications and was not studied in the previous papers [2–4].

In the present paper, we show, first of all, that the space of functions corresponding to a fixed pseudomomentum can be used as the space of the functions describing the relative motion of the system (with...
respect to the center-of-mass). This seems to be unexpected and paradoxical because no separation of the motion of the center-of-mass is carried out—this is, in principle, impossible (unless all of the particles are identical). We simultaneously establish that the operator $H$ under study can be brought to the form $H_0$ in which it depends only on the relative coordinates and, moreover, its “differential part” involves only the projections of the gradient onto the hyperplane of the relative motion.

The above-mentioned properties made it possible to apply the geometric methods developed in [2–4] to study the operator $H_0$ and obtain the following results.

I. The boundary of the essential spectrum of the operator $H_0$ is determined in terms of the usual statement of the Hunziker-Van Winter-Zhislin theorem. (A conceptually equivalent but more complicated statement of this theorem was proved in [1] by another method.)

II. Theorems on the structure of the discrete spectra of two-particle neutral systems with a fixed pseudomomentum are proved. Namely, conditions for the finiteness and infiniteness of the discrete spectrum are found and asymptotic formulas with estimates for the remainder are derived. These apply, in particular, to the Hamiltonian of the hydrogen atom. Note that the only result for the spectrum of the operator $H_0$ that was known before was confined to the existence of at least one eigenvalue in the case of a negative interaction potential [1].

The discrete spectrum of systems with three or more particles will be considered in a separate article because their study encounters some serious additional difficulties. These relate to the investigation of the properties of the operators for the partitions of the original system that define the boundary of the essential spectrum of the operator $H_0$.

1. Fixed pseudomomentum. Statement of the problem

1.1. Consider an arbitrary $n$-particle quantum system in a homogeneous magnetic field directed along the $z$ axis.

Let $r_i = (x_i, y_i, z_i)$, $m_i$, and $e_i$ be the radius vector, the mass, and the charge of the $i$th particle, respectively. We write $r = (r_1, \ldots, r_n)$,

$$M = \sum_{i=1}^{n} m_i, \quad Q = \sum_{i=1}^{n} e_i,$$

where $\rho = (\rho_1, \ldots, \rho_n)$, and $\rho_i = (x_i, y_i)$. After the motion of the center-of-mass in the direction of the $z$ axis is separated, i.e., the Hamiltonian of the system is restricted to the subspace of the functions defined on the set

$$R_{03} = \left\{ r \left| \sum_{i=1}^{n} m_i z_i = 0 \right. \right\},$$

we can represent the energy operator of the system in the form

$$H = T_\perp + T^{03}_\perp + V(r),$$

(1.1)

where

$$T_\perp = \sum_{j=1}^{n} T_j, \quad T_j = \left( \frac{1}{i} \nabla_{j\perp} - c_j B_j \right)^2 \frac{1}{m_j},$$

$$\nabla_{j\perp} = \nabla_{j\perp}(\rho) = \left( \frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j} \right).$$