Wavelet Sets in $\mathbb{R}^n$

Xingde Dai, David R. Larson, and Darrin M. Speegle

ABSTRACT. A congruency theorem is proven for an ordered pair of groups of homeomorphisms of a metric space satisfying an abstract dilation-translation relationship. A corollary is the existence of wavelet sets, and hence of single-function wavelets, for arbitrary expansive matrix dilations on $L^2(\mathbb{R}^n)$. Moreover, for any expansive matrix dilation, it is proven that there are sufficiently many wavelet sets to generate the Borel structure of $\mathbb{R}^n$.

A dyadic orthonormal (or orthogonal) wavelet is a function $\psi \in L^2(\mathbb{R})$, (Lebesgue measure), with the property that the set

$$\{2^n \psi(2^n t - l): n, l \in \mathbb{Z}\}$$

is an orthonormal basis for $L^2(\mathbb{R})$ (see [1, 2]). For certain measurable sets, $E$, the normalized characteristic function $\frac{1}{\sqrt{2\pi}} x_E$ is the Fourier transform of such a wavelet. There are several characterizations of such sets (see [3] chapt. 4, and independently [5]). In [3] they are called wavelet sets. In [5, 6, 7] they are the support sets of MSF (minimally supported frequency) wavelets.

Dilation factors on $\mathbb{R}$ other than 2 have been studied in the literature, and analogous wavelet sets corresponding to all dilations $> 1$ are known to exist ([3], Example 4.5, part 10). Matrix dilations (for real expansive matrices) on $\mathbb{R}^n$ have also been considered in the literature, usually for a "multi-

notion of wavelet. The translations involved are those along the coordinate axes. The purpose of this article is to prove a general-principle type of result that shows, as a corollary, that analogous wavelet sets exist (and are plentiful) for all such dilations. In particular, "single-function" wavelets always exist. This appears to be new. Theorem 1 seems to belong to the mathematics behind wavelet theory. For this reason we prove it in a more abstract setting than needed for our wavelet results. Essentially, it is a dual-dynamical system congruency principle. The general proof is no more difficult than that for $\mathbb{R}^n$.

We point out that the wavelets we obtain, which are analogs of Shannon’s wavelet, need not satisfy the regularity properties often desired (see [8]) in applications.

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there are measurable partitions \( \{ E_g : g \in G \} \) and \( \{ F_g : g \in G \} \) of \( E \) and \( F \), respectively, such that \( F_g = g(E_g) \) for each \( g \in G \), modulo \( m \)-null sets.

If \( r > 0 \) and \( y \in X \), we write \( B_r(y) := \{ x \in X : \| x - y \| < r \} \), and abbreviate \( B_r := B_r(0) \).

We will say that \((D, T)\) is an abstract dilation-translation pair if (1) for each bounded set \( E \) and each open set \( F \) there are elements \( \delta \in D \) and \( \tau \in T \) such that \( \tau(E) \subseteq \delta(F) \), and (2) there is a fixed point \( \theta \) for \( D \) in \( X \) which has the property that if \( N \) is any neighborhood of \( \theta \) and \( E \) is any bounded set, there is an element \( \delta \in D \) such that \( \delta(E) \subseteq N \).

**Theorem 1.**

Let \( X, B, m, D, T \) be as above, with \((D, T)\) an abstract dilation-translation pair, and with \( \theta \) the \( D \)-fixed point as above. Let \( E \) and \( F \) be bounded measurable sets in \( X \) such that \( E \) contains a neighborhood of \( \theta \), and \( F \) has nonempty interior and is bounded away from \( \theta \). Then there is a measurable set \( G \subseteq X \), contained in \( \bigcup_{\delta \in D} \delta(F) \), which is both \( D \)-congruent to \( F \) and \( T \)-congruent to \( E \).

**Proof.** We will use the term "\( D \)-dilate" to denote the image of a set \( \Omega \) under an element of \( D \), and "\( T \)-translate" for the image of \( \Omega \) under an element of \( T \).

We will construct a disjoint family \( \{ G_{ij} : i \in \mathbb{N}, j \in \{1, 2\} \} \) of measurable sets whose \( D \)-dilates form a partition \( \{ F_{ij} \} \) of \( F \) and whose \( T \)-translates form a partition \( \{ E_{ij} \} \) of \( E \), modulo \( m \)-null sets. Then \( G = \bigcup_{i,j} G_{ij} \) will clearly satisfy our requirements. The \( i \)th induction step will consist of constructing \( G_{11} \) and \( G_{12} \).

Let \( \{ \alpha_i \} \) and \( \{ \beta_i \} \) be sequences of positive constants decreasing to 0. Let \( N_1 \subseteq E \) be a ball centered at \( \theta \) with radius \( \alpha_1 \) such that \( m(E \setminus N_1) > 0 \). Let \( E_{11} = E \setminus N_1 \).

Observe that we may choose \( \delta_1 \in D \), \( \tau_1 \in T \), so that \( (\delta_1^{-1} \circ \tau_1)(E_{11}) \) is a subset of \( F \) whose relative complement in \( F \) has a nonempty interior. This is possible since the interior of \( F \) is nonempty, and there is a \( \delta_1 \)-dilate of \( F \) which contains a ball large enough to contain some \( \tau_1 \)-translate of \( E \) with ample room left over. Now set \( F_{11} := (\delta_1^{-1} \circ \tau_1)(E_{11}) \). (In this context, clearly we may choose \( \delta_1 \) and \( \tau_1 \) such that, in addition, the \( \tau_1 \)-translate of \( E \) is disjoint from any prescribed bounded set — a fact that will be useful in the second and subsequent steps.)

Let \( G_{11} := \tau_1(E_{11}) = \delta_1(F_{11}) \). Since \( \delta_1 \) is a homeomorphism of \( X \) which fixes \( \theta \), \( G_{11} \) is bounded away from \( \theta \) since \( F_{11} \) is. Let \( F_{12} \) be a measurable subset of \( F \) of positive measure, disjoint from \( F_{11} \), such that the difference \( F \setminus (F_{11} \cup F_{12}) \) has a nonempty interior and measure \( < \beta_1 \). Choose \( \gamma_1 \in D \) such that \( \gamma_1(F_{12}) \) is contained in \( N_1 \) and is disjoint from \( G_{11} \). Set \( E_{12} := \gamma_1(F_{12}) \), and set \( G_{12} := E_{12} \). The first step is complete.

For the second step, note that since \( F \) is bounded away from \( \theta \), \( N_1 \setminus E_{12} \) contains a ball \( N_2 \) centered at \( \theta \) with radius \( < \alpha_2 \) such that \( N_1 \setminus (E_{12} \cup N_2) \) has positive measure. Let

\[
E_{21} := N_1 \setminus (E_{12} \cup N_2) = E \setminus (E_{11} \cup E_{12} \cup N_2).
\]

Choose \( \delta_2 \in D \), \( \tau_2 \in T \), using similar reasoning to that used above, such that \( (\delta_2^{-1} \circ \tau_2)(E_{21}) \) is a subset of \( F \setminus (F_{11} \cup F_{12}) \) whose relative complement in \( F \setminus (F_{11} \cup F_{12}) \) has a nonempty interior, and for which \( \tau_2(E_{21}) \) is disjoint from \( G_{11} \) and \( G_{12} \). Let \( F_{21} := (\delta_2^{-1} \circ \tau_2)(E_{21}) \), and let \( G_{21} := \tau_2(E_{21}) \).

Choose a measurable subset \( F_{22} \subset F \) of positive measure disjoint from \( F_{11} \), \( F_{12} \), \( F_{21} \) such that \( F \setminus (F_{11} \cup F_{12} \cup F_{21} \cup F_{22}) \) has a nonempty interior and measure \( < \beta_2 \). Noting that \( G_{11} \), \( G_{12} \), \( G_{21} \) are bounded away from \( \theta \), choose \( \gamma_2 \in D \) such that \( \gamma_2(F_{22}) \) is contained in \( N_2 \) and is disjoint from \( G_{11} \), \( G_{12} \), \( G_{21} \). Set \( E_{22} := \gamma_2(F_{22}) \), and let \( G_{22} := E_{22} \).

Now proceed inductively, obtaining disjointed families of sets of positive measure \( \{ E_{ij} \} \) in \( E \), \( \{ F_{ij} \} \) in \( F \), and \( \{ G_{ij} \} \), such that

\[
\delta_1^{-1}(G_{11}) = E_{11}, \quad G_{12} = E_{12}, \quad G_{21} = \delta_1^{-1}(G_{11}) = F_{11},
\]

\[
\gamma_1^{-1}(G_{12}) = F_{12}, \quad \text{for } i = 1, 2, \ldots \text{ and } j = 1, 2.
\]