ON THE ISOMORPHISM OF WREATH PRODUCTS OF GROUPS

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The well-known Neumann theorem on the isomorphism of standard wreath products is generalized to the wreath products of an arbitrary transitive permutation group and an abstract group.

In this paper, we generalize the well-known Neumann theorem [1] on the isomorphism of standard wreath products of groups to the wreath products of an arbitrary transitive permutation group and an abstract group. Note also papers [2, 3] where similar problems are considered.

Let \( W = (G, X) * H \) and \( W_1 = (B, Y) * L \) be Cartesian (direct) wreath products of permutation groups and abstract groups. We denote their bases by \( F \) and \( V \), respectively.

Theorem. Let \( \mu : W \to W_1 \) be an isomorphism of Cartesian (direct) wreath products. Then only one of following statements is true:

(i) \( F \sim V \); in this case, \( H \equiv L \) and there exists an automorphism \( \gamma \in \text{Aut} W_1 : G^{\pi_1} = B \).

(ii) \( F \supset V \); in this case, we can continue decomposing \( W \) and \( W_1 \) into the Cartesian (direct) wreath products, i.e., \( (B, Y) = (B_1, Y_1) * (R, Y_0) \) and \( H \equiv (R, Y_0) * L \), where \( (B_1, Y_1) \) and \( (R, Y_0) \) are transitive permutation groups. By using the associativity of the operation of wreath product of a permutation group, this case can be reduced to case (i).

(iii) \( F \subset V \). This case can be reduced to case (ii) by the substitution \( \mu \to \mu^{-1} \).

(iv) \( F \not\subset V \) and \( F \not\supset V \); in this case, \( H \equiv L \) is a dihedral group of special type [1] and the following decompositions into Cartesian (direct) wreath products take place: \( (G, X) = (G^1, X^1) * C_2 \) and \( (B, Y) = (B^1, Y^1) * C_2^2 \); here, \( C_2^2 \) is a second-order cyclic group. By permutation of the brackets in \( W \) and \( W_1 \), this case can be reduced to case (i).

Proof. We introduce the following notation:

\[
\varepsilon_{i, l}(y) = \begin{cases} 
  l \in L & \text{if } y = i \in Y; \\
  e \text{ (the identity element)} & \text{if } t \neq l.
\end{cases}
\]

We also set \( Q_x = \{ \varepsilon_{x, h} \mid h \in H \} \equiv H \) and \( P_y = \{ \varepsilon_{y, l} \mid l \in L \} \equiv L \).

The subgroups \( Q_x, x \in X, \) and \( P_y, y \in Y, \) generate the bases of wreath products. Let \( x_0 \in X \) and \( y_0 \in Y \) be arbitrary points and let \( G_0 \) and \( B_0 \) be their stabilizers in \( G \) and \( B \), respectively. We set \( Q = Q_{x_0} \) and \( P = P_{y_0} \) and denote the canonical homomorphisms \( W \to G \) and \( W_1 \to B \) by \( \pi_1 \) and \( \pi \), respectively. We have
This relation must hold for any $w \in W$ such that $w^{-1} \in G_0$.

Consider case (i) of the theorem. If the basis is transformed into the basis, then the image of the group $G$ is a certain complement to $V$, i.e., $W = V \Lambda G^\mu$. According to [4], each complement is characterized by the homomorphism $\theta : B_0 \to L$ up to conjugacy.

Lemma 1. If $\theta$ is the homomorphism corresponding to the complement $G^\mu$, then $\text{Im} \theta \leq Z(L)$.

Proof. Assume that $G^\mu = \{ b V_b | b \in B \}$ and the corresponding homomorphism $\theta$ is defined as follows:

$$b_0^\mu = v_0(y_0)$$

for any $b_0 \in B_0$. Let $f_l(x)$ be the preimage of the function $\epsilon_{x_l,l}$ and let $\psi$ be an arbitrary function from $F$. Since $(\psi^\mu)\psi^{-1}(y_0) = e \forall b_0 \in B_0$, we have $[\epsilon_{y_0,l}, b_0(\psi^\mu)\psi^{-1}] = e \forall l \in L$, $b_0 \in B_0$, $\psi \in F$. Let $w^\mu = b_0$, where $w = g_1 \varphi_1 \in W$. Since $b_0^\mu b_0(\psi^{-1})^{-1} = (\psi w \psi^{-1})^\mu$, by applying $\mu^{-1}$ to the commutator equality, we obtain $[f_l, \psi w \psi^{-1}] = e$ or $[f_l, g_1 \varphi_1 \psi^{-1}] = e$. Let $z \in X$ be an arbitrary point. If $g_1 z \neq z$, then we can choose $\psi$ so that $\psi \varphi_1 \psi^{-1}(z) = e$. Then the previous equality yields $f_l(g_1 z) = f_l(z)$. If $g_1 z = z$, this equality is also true and, hence, $[f_l, g_1] = e$ for any $l \in L$. Since $(g_1 \varphi_1)^\mu = b_0$, we have $g_1^\mu = b_0 \varphi_{b_0}$, where $\varphi_{b_0} = (\varphi_1^\mu)^{-1}$. By applying $\mu$ to the last commutator equality, we obtain $[\epsilon_{y_0,l}, b_0 \varphi_{b_0}] = e$ for any $l \in L$. Therefore, $\varphi_{b_0}(y_0) \in Z(L)$ for any $b_0 \in B_0$. The lemma is proved.

It follows from [4] that we can construct an automorphism $\gamma \in \text{Aut} W_1$ such that $B^\gamma = G^\mu$, $\nu^\gamma = \nu \forall \nu \in V$. By passing to the isomorphism $\mu_1 = \mu \gamma^{-1}$, we obtain $G^\mu = B$, $F^\mu = V$. The proof of the isomorphism $H \cong L$ given in [1] for standard wreath products can be literally carried over to the general case.

Let us present several assertions concerning isomorphisms of type (ii)–(iv) which will be necessary in what follows. Assume that $\epsilon_{x_0,h} = b_h \nu_h$, where $h \in H$, $\nu_h \in V$, and $\{ b_h \nu_h | h \in H \} = G^\mu \nu$.

By applying $\mu$ to relation (1), we obtain the equality

$$[w b_h \nu_h w^{-1}, b_{h_1} \nu_{h_1}] = e$$

for all $h, h_1 \in H$ and arbitrary $w \in W_1$ such that

$$w^{\mu^{-1}} \varnothing \not\in G_0.$$ (3)

By substituting $w = q^{-1} p(y)$, $q \in B$, $p(y) \in V$, in (2), we get

$$b_h^q b_{h_1} = b_{h_1} b_h^q,$$

$$\left( p^{b_h} \nu_h p^{-1} \right)^{q b_h} \nu_{h_1} = \nu_{h_1}^q \left( p^{b_h} \nu_h p^{-1} \right)^q.$$ (5)

Lemma 2. If $\mu$ is a homomorphism such that $F^\mu \nu \neq \{ e \}$, then there exists a function $p(x) \in F$ such that $p(x)^\mu \not\in B_0$. 

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