A NOTE ON REGULARITY AND EXISTENCE OF SOLUTIONS FOR A CLASS OF NON–UNIFORMLY DEGENERATE ELLIPTIC EQUATIONS*

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Abstract. In this paper we consider the Dirichlet problems of a non-uniformly degenerate elliptic equations of the form

\[ \text{div} A(x, Du) + B(x) = 0 \quad \text{in} \quad \Omega \]  

(0.1)

whose prototype is

\[ \text{div}(b(x)|Du|^\alpha-1 Du + (1 - b(x))|Du|^{\beta-1} Du) = 0 \quad \text{in} \quad \Omega, \]  

(0.2)

where \( \Omega \subset \mathbb{R}^N \) is a bounded domain, \( 0 \leq b(x) \leq 1, \) \( 0 < \alpha \leq \beta. \) We establish that if \( A \) and \( B \) are under some structure conditions and \( 0 < \alpha \leq \beta < \max \{ \frac{N+1}{N} \alpha + \frac{1}{N}, \alpha + 1 \}, \) then there exists a \( C^{1+a} \)-solution of (0.1) associated with the Dirichlet boundary data.

Key Words. Elliptic Equation, Non-uniformly Degenerate.

AMS Subject Classifications. 35D05,35J70.

1. Introduction

There is an extensive literature on the existence and regularity of weak solutions to uniformly degenerate elliptic equations, see \([1,2,5,8,9,10]\).

In \([7]\), Marcellini studied the non-uniformly non-degenerate elliptic equations of the form

\[ \text{div} A(x, Du) + B(x) = 0 \quad \text{in} \quad \Omega, \]  

(1.1)

where \( \Omega \subset \mathbb{R}^N \) is a bounded domain and

\[ A(x, p) = (A^1(x, p), \cdots, A^N(x, p)) \]
satisfies
\[ \lambda(1 + |p|)^{\alpha-1} |\xi|^2 \leq \frac{\partial A^i}{\partial p_j}(x, p)\xi_i\xi_j \leq \Lambda(1 + |p|)^{\beta-1} |\xi|^2 \quad (1 \leq \alpha \leq \beta). \tag{1.2} \]

In addition, when \( \alpha, \beta \) and \( A \) satisfy
\[ \beta < \frac{N + 2}{N} \alpha + \frac{2}{n} \quad \text{and} \quad |A_x(x, p)| \leq c(1 + |p|)^{\frac{\alpha + \beta}{2}}, \tag{1.3} \]

Marcellini obtained the local \( \|Du\|_{L^\infty} \)-estimate in terms of \( \|Du\|_{L^{\alpha+1}} \), \( \alpha, \beta, \lambda \) and \( \Lambda \), therefore established the existence of Lipschitz continuous weak solutions of (1.1) for Dirichlet problems.

Recently the author [4] considered the non-uniformly degenerate elliptic equations of the form (1.1) with \( A(x, 0) = 0 \) and
\[ \min\{|p|^{\alpha-1}, |p|^{\beta-1}\} |\xi|^2 \leq \frac{\partial A^i}{\partial p_j}(x, p)\xi_i\xi_j \leq a_0(|p|^{\alpha-1} + |p|^{\beta-1}) |\xi|^2 \tag{1.4} \]

for all \( p \in \mathbb{R}^N \setminus \{0\}, \xi \in \mathbb{R}^N \),
\[ |A(x, p)| + |A_x(x, p)| \leq a_0(1 + |p|^\beta), \tag{1.5} \]
\[ 0 < \alpha \leq \beta < \max\{\frac{N + 1}{N} \alpha + \frac{1}{N}, \alpha + 1\}. \tag{1.6} \]

The existence of \( C^1 \)-continuous weak solutions of (1.1) for Dirichlet problems was established in [4].

The purpose of this work is to obtain \( C^{1+\sigma} \)-weak solutions of (1.1) for Dirichlet problems when \( A(x, p) \) satisfies
\[ [b(x)|\xi|^{\alpha-1} + (1 - b(x))|p|^{\beta-1}] |\xi|^2 \leq \frac{\partial A^i}{\partial p_j}(x, p)\xi_i\xi_j \leq a_0[b(x)|p|^{\alpha-1} + |p|^{\beta-1}] |\xi|^2 \tag{1.7} \]
\[ |A(x, p)| \leq a_0(b(x)|p|^\alpha + |p|^\beta), \quad |A_x(x, p)| \leq a_0(1 + |p|^\beta), \tag{1.8} \]

where \( 0 \leq b(x) \leq 1, \alpha \) and \( \beta \) are related by (1.6).

Our case include the following

**Example 1.1.**
\[ \sum_{i=1}^{N} \frac{\partial}{\partial x_i} (b(x)|Du|^{\alpha-1}u_{x_i} + (1 - b(x))|Du|^{\beta-1}u_{x_i} + c_i(x)|u_{x_i}|^{\alpha_i-1}u_{x_i} = 0, \]

where \( b(x) \) and \( c_i(x) \) are Lipschitz functions with \( 0 \leq b(x) \leq 1, c_i(x) \geq 0, 0 < \alpha \leq \alpha_1 \leq \cdots \leq \alpha_N \leq \beta, \alpha_1 \geq 1. \)