ON APPROXIMATION OF OPTIMAL STOPPING
OF BAYESIAN SEQUENTIAL TEST FOR A NORMAL MEAN*

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Abstract. In this paper, we present a simple and direct approach in which super-
martingales are used to approximate the optimal stopping sets associated with the
Bayesian sequential test for normal population means. Several conclusions are given.

The sequential tests for normal population means often arise from or are created
through approximations or transformations in practice. A standard case is as fol-
low. Let \( X_1, X_2, X_3, \ldots \) be a sequence of mutually independent observations from a
normal population \( X \) with distribution \( N(\theta, \sigma^2) \). We want to test

\[ H_0 : \theta \leq 0 \quad \text{versus} \quad H_1 : \theta > 0. \]

Suppose that the cost of a wrong decision is \( r(\theta) = k|\theta| \), and the cost of sampling
by time \( n \) is \( c_0 n \). When does the parameter \( \theta \) have the prior distribution \( N(\theta_0, \sigma_0^2) \).
and what is the optimal Bayes sequential strategy (provided \( \sigma^2, \sigma_0^2, \theta_0, k \) and \( c_0 \) are
known)?

The numerical optimal solution of the problem can directly be obtained via back-
ward induction, but analytic approximation is still necessary [2]. A classical analytic
approximation is related to the solution of a heat equation with free boundary.

In Section 2, we establish a proposition based on which one can select certain su-
permartigales to encompass the approximation of the optimal stopping sets directly.
This proposition is applied to the Bayesian sequential test for a normal population
mean. Certain approximate optimal stopping sets are given in Section 3. Finally we
compare our approximations with previous ones.

1. Preliminaries

Without loss of generality, we assume that \( \sigma^2 = 1, \theta_0 = 0 \) and \( k_0 = 0 \). Let
\( r_0 = \sigma_0^{-2}, r_n = r_{n-1} + 1, k_n = k_{n-1} + X_n \), and \( \theta_n = k_n/r_n \) for \( n \geq 1 \). A random
variable \( \tau \) taking values in \( \{0, 1, \ldots, \infty\} \) is called a stopping rule if \( P(\tau < \infty) = 1 \)
and \( (\tau = n) \in F_n \), where \( F_n \) is the \( \sigma \)-algebra generated by \( X_1, \ldots, X_n \) for \( n \geq 1 \).

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In the remainder of this paper, the collection \( \{F_n\} \) will not be mentioned when it occurs. Write
\[
\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}x^2\} \quad \text{and} \quad \Phi(x) = \int_{-\infty}^{x} \phi(t)dt.
\]
We then have
\[
P(\theta|\theta_n) = \sqrt{r_n} \phi(\sqrt{r_n}(\theta - \theta_n)). \tag{1.1}
\]
Hence \( \{\theta_n\} \) is a Markov process. Now the minimum posterior Bayesian risk of sampling at the nth time is
\[
\min\{ E(\theta^+|\theta_n), E(\theta^-|\theta_n) \} + cn = \frac{1}{2}\left[ E(|\theta| |\theta_n) - |\theta_n| \right] + cn
\]
with \( \theta^+ = \max(\theta, 0) \) and \( \theta^- = -\min(\theta, 0) \). The optimal Bayesian strategy is to find a stopping rule which minimizes the expected risk, or equivalently, maximizes the expectation of the reward
\[
R_n \equiv R_n(\theta_n) = |\theta_n| - 2cn, \quad n = 1, 2, \ldots \tag{1.2}
\]
with \( c = c_0/k \). Such a stopping rule, said to be optimal, exists in the situation we consider and its proof will be given in the next section. For a similar situation, see Wu [6].

Let \( E(\cdot |\theta_n) \) denote the posterior expectation related to \( P(\cdot |n, \theta_n) \). We define the optimal stopping reward by
\[
S_n \equiv S_n(\theta_n) = \sup E(R_\tau |\theta_n), \quad n \geq 1,
\]
where the supremum is taken over all stopping rules \( \tau \) such that \( \tau \geq n \). \( S_n \) can equivalently be determined by the dynamic equation [3]
\[
S_n(\theta_n) = \max\{R_n(\theta_n), E(S_{n+1}|\theta_n)\}. \tag{1.3}
\]
A pair \( (n, \theta_n) \) is called an optimal stopping point if \( S_n(\theta_n) = R_n(\theta_n) \), and the set \( \tilde{D}_\theta \) consisting of all optimal stopping points is called the optimal stopping set. Since the time \( \tilde{\tau} \) first entering into \( \tilde{D}_\theta \) is just an optimal stopping rule [3], it suffices to focus our attention on the approximation of \( \tilde{D}_\theta \).

The state space of \( \{\theta_n\} \) is the real line. A pair \( (m, \theta_m) \) is called a successor of \( (n, \theta_n) \) if the process can arrive at \( \theta_m \) from \( \theta_n \) when one samples from time \( n \) to time \( m \). By a truncated time we mean a fixed time \( N \) at which \( S_N = R_N \) almost surely. In addition, by a \( M \)-supermartingale \( \{g_m\} \), we mean that \( g_m \equiv g_m(\theta_m) \) satisfies the relationship \( E(g_{m+1}|\theta_m) \leq g_m(\theta_m) \) for \( m \geq 1 \).