A DIRECT METHOD IN OPTIMAL PORTFOLIO AND CONSUMPTION CHOICE*

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Abstract. In this paper, we use a direct method to solve the optimal portfolio and consumption choice problem in the security market for a specific case, in which the utility function is of a given homogenous form, i.e. the so-called CRRA case. The idea comes from the completion technique ever used in LQ optimal control.

1. Asset Equation and Optimal Strategy Problem

In this introductory section, we will give a brief setting of our problem. We first introduce the classical asset equation in the security market, and then we state the optimal consumption/investment problem and the canonical method for solving it. Finally we state the important CRRA case which we want to consider in the paper.

Suppose the investor has the choice of only two different investments. One of the assets is called bond whose price \( p_0(t) \) is assumed to satisfy an ordinary differential equation

\[
dp_0(t) = r_t p_0(t)dt,
\]

where \( r_t \) is the interest rate at time \( t \). The other asset is risky, called stock. We assume that the price \( p_1(t) \) of this asset admits the following stochastic differential equation

\[
dp_1(t) = u_t p_1(t)dt + \sigma_t p_1(t)dB_t,
\]

where \( u_t \) represents the instantaneous expected rate of return, \( \sigma_t \) the instantaneous volatility; \( B_t \) is an one-dimensional Brownian motion on some given probability space \((\Omega, \mathcal{F}, P)\), \( \mathcal{F}_t \) is assumed to be the natural filtration of \( B_t \).

Here \( r_t, u_t \) and \( \sigma_t \) are assumed to be deterministic and bounded uniformly in \( t \in [0, T] \); \( \sigma_t > 0 \) and \( \sigma_t^{-1} \) is also bounded. Furthermore, it is natural to assume \( r_t \sim u_t \).

We denote by \( \pi_t \) the amount invested in the stock and by \( c_t \) the consumption process of the investor. We assume that \((c_t, \pi_t) \in M^2(0,T) \times M^2(0,T)\). At each moment \( t \), the investor may choose freely his consumption and the number of dollars

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invested in the risky asset. The wealth of the investor who starts with some initial endowment \( w_0 > 0 \) thus is modeled by

\[
dw_t = [r_t w_t - c_t + (u_t - r_t)\pi_t] \, dt + \sigma_t \pi_t \, dB_t, \\
w(0) = w_0.
\]

Suppose the investor wants to maximize the expected utility of the wealth

\[
J(c, \pi) = E\left\{ \int_0^T g(c_t, t) \, dt + h(w_T, T) \right\}
\]

at some future time \( T \) by making appropriate consumption/investment decisions at each time \( t \in [0, T] \). \((c_t, \pi_t) \in M^2(0, T) \times M^2(0, T)\) is called an optimal strategy if it attains the maximum of \( J(c, \pi) \).

The above problem is obviously a stochastic optimal control problem. The classical method for solving it is the dynamical programming method, from which the corresponding HJB equation can be obtained. Therefore one can represent the optimal consumption/investment in an abstract form following the HJB equation. The interested reader can see [1] and [2] for detail.

As well known, one can only for special cases write the optimal strategies in state feedback form by solving the corresponding HJB equation. One of the most important cases is the so-called CRRA case, where

\[
g(c, t) = Le^{-\beta t} \frac{c^{1-R}}{1-R} \quad \text{and} \quad h(w, T) = K \frac{w^1-R}{1-R}, R \in (0, 1).
\]

The dynamical programming principle is rather demanding and relatively profound for the reader who is interested in investment decisions in the security market but may not be familiar with this mathematical field. For the specific CRRA case mentioned above, we use a relatively simpler method to solve it in the paper. Our method is direct and easy to understand. The idea comes from the completion technique ever used for solving the celebrated LQ problem in optimal control.

2. A Direct Method

We first rewrite the asset equation and restate the optimal strategy problem. As we have mentioned in Section 1, the wealth of the investor satisfies

\[
dw_t = [r_t w_t - c_t + (u_t - r_t)\pi_t] \, dt + \sigma_t \pi_t \, dB_t, \\
w(0) = w_0 > 0. \tag{1}
\]

The investor wants to seek a pair \((c, \pi) \in M^2(0, T) \times M^2(0, T)\) to maximize his utility

\[
J(c, \pi) = E\left\{ \int_0^T Le^{-\beta t} \frac{c^{1-R}}{1-R} \, dt + K \frac{1}{1-R} w_T^{1-R} \right\}. \tag{2}
\]