Systems Analysis

Using Categorical Methods
In Computer Science

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Some categorical methods used in developing computer programs are considered. Relevant theorems are proved.

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A modern trend in the development of computer science is the construction of algorithms and other objects (related to computations) by categorical methods. At the same time, as is well known, proofs of statements in the theory of categories are largely processes of symbol manipulation. Therefore, these processes can be described as programs of the highest abstraction level owing to their separation from data. The authors accumulate some experience in using categorical methods in various domains of computer science. In particular, we proposed a formalization of different aspects of information technologies, including methods of design of program systems [1, 2], constructed a model of incompleteness in a topos, proved a number of theorems on incompleteness and solvability [3], etc.

This article deals with problems of solution of equations and construction of an algorithm of finding the most common unifier on the basis of general categorical constructions. An approach is proposed that provides provably correct construction of algorithms by formation of systems of categorical axioms. Objects are defined that make it possible to construct analogs of well-known classical numerical systems in a topos. Some illustrative examples are given.

Categories of Term Rewriting. Let us define a category whose objects are sets and whose arrows are substitutions of terms [4].

Let an operator collection represent a set of operator symbols indexed by their dimension. If Ω is such an operator collection, then the set of operators of arity n is denoted by Ω_n. The set of terms TΩ(X) over the operator collection Ω in a set X is recursively defined as follows:

\[ x \in X \Rightarrow (x) \in TΩ(X), \]
\[ \sigma \in Ω_n \Rightarrow t_1, t_2, \ldots, t_n \in TΩ(X) \Rightarrow \sigma(t_1, t_2, \ldots, t_n) \in TΩ(X). \]

We call an arbitrary mapping \( w: X \rightarrow TΩ(Y) \) a substitution of terms over a set Y for the elements of the set X. For example, if \( X = \{x_1, x_2, x_3\} \), \( Y = \{x, y, z\} \), and \( f \) and \( g \) are, respectively, monadic and binary operators, then a substitution \( w: X \rightarrow TΩ(Y) \) can be specified by the relations \( w(x_1) = g(x, y) \), \( w(x_2) = g(f(x), g(x, y)) \), and \( w(x_3) = f(f(z)) \) or the substitutions \( x_1 \mapsto g(x, y), x_2 \mapsto g(f(x), g(x, y)), \) and \( x_3 \mapsto f(f(z)) \).

It is easily seen that the relations

\[ f((x)) = f(x), \quad x \in X, \]
\[ f(\sigma(t_1, t_2, \ldots, t_n)) = \sigma(f(t_1), f(t_2), \ldots, f(t_n)) \]

allows one to apply substitutions to terms. The composition of substitutions is defined as usual, i.e., \( (fg)(x) = f(g(x)) \), and the identical substitution is defined as \( i(x) = (x) \).
Hence, a category $\mathcal{T}_\Omega$ is defined whose objects are sets and whose arrows are substitutions of terms. This category is an example of the Kleisli category [5].

2-Categories of Term Rewriting and Solution of Equations. Rewriting rules are oriented equalities inducing the structure of a 2-category on the category $\mathcal{T}_\Omega$.

Let us illustrate the aforesaid by an example. Assume that the operator collection $\Omega$ includes a constant $e$, a monadic operator $in$, and a binary operator $\ast$. We also assume that the pair of arrows $< w: X \rightarrow Y, v: X \rightarrow Y >$ of the category $\mathcal{T}_\Omega$ determines a 2-arrow of this category. In particular, the pair of arrows

\[
\begin{align*}
    z \mapsto x \ast y \\
    \{z\} \hspace{1cm} \{x, y\} \\
    z \mapsto y \ast x
\end{align*}
\]

specifies a 2-arrow $\text{commute}$.

Let us define a composition of this 2-arrow and the substitution $h: \{x, y\} \rightarrow \{u, v\}$ specified by the relations $h(x) = in(u)$ and $h(y) = v$. To this end, we construct the compositions $(z \mapsto x \ast y)h$ and $(z \mapsto y \ast x)h$ in the category $\mathcal{T}_\Omega$. Then, the 2-arrow $\text{commute}h$ is defined by the pair of arrows

\[
\begin{align*}
    z \mapsto in(u) \ast v \\
    \{z\} \hspace{1cm} \{u, v\} \\
    z \mapsto v \ast in(u)
\end{align*}
\]

of the category $\mathcal{T}_\Omega$ and is called the right composition of the 2-arrow $\text{commute}$ and the substitution $h$. In general, the right composition of a 2-arrow $< w: X \rightarrow Y, v: X \rightarrow Y >$ and a substitution $h: Y \rightarrow Z$ is the 2-arrow $< wh: X \rightarrow Z, vh: X \rightarrow Z >$.

The left composition of a 2-arrow and a substitution is similarly defined. Namely, if a substitution $g: \{z'\} \rightarrow \{z\}$ specified by the relations $g(z') = z' \ast in(z)$ is given, then the 2-arrow $g \circ \text{commute}$ is specified by the pair of arrows

\[
\begin{align*}
    z' \mapsto (x \ast y) \ast in(x \ast y) \\
    \{z'\} \hspace{1cm} \{x, y\} \\
    z' \mapsto (y \ast x) \ast in(y \ast x)
\end{align*}
\]

of the category $\mathcal{T}_\Omega$ and is called the left composition of the 2-arrow $\text{commute}$ and the substitution $g$. In general, it is the arrow $< gw: Z \rightarrow Y, gv: Z \rightarrow Y >$, where $g: Z \rightarrow X$.

By definition, the vertical composition of two 2-arrows $\alpha = < w: X \rightarrow Y, v: X \rightarrow Y >$ and $\alpha' = < w': X \rightarrow Y, r: X \rightarrow Y >$ is the arrow $\alpha \circ \alpha' = < w w': X \rightarrow Z, vv': Y \rightarrow Z >$. By definition, the horizontal composition of two 2-arrows $\alpha = < w: X \rightarrow Y, v: X \rightarrow Y >$ and $\beta = < w: Y \rightarrow Z, v: Y \rightarrow Z >$ is the arrow $\alpha \circ \beta = < w w': X \rightarrow Z, vv': X \rightarrow Y >$.

We now turn to concrete 2-arrows specified by pairs of arrows of the category $\mathcal{T}_\Omega$, namely,

\[
\begin{align*}
    \alpha: \{z\} & \rightarrow \{z\} \\
    z \mapsto e \ast z \\
    z \mapsto z \ast e \\
    \alpha': \{z\} & \rightarrow \{z\} \\
    z \mapsto z \ast e \\
    z \mapsto z \\
    \beta: \{z\} & \rightarrow \{z\} \\
    z \mapsto z \ast z \\
    z \mapsto z \\
\end{align*}
\]