EXPANSION OF CLASSICAL NONCOOPERATIVE EQUILIBRIUM AND PROGRAM DIFFERENTIAL GAMES

É. R. Smol'yakov

A generalization of the classical noncooperative equilibrium is proposed. The new equilibrium exists in a broader class of games and problems of accepting proposals than the classical equilibrium. Applications of this equilibrium to program differential games are considered.

Keywords: noncooperative equilibrium, program differential games, Roos–Nash equilibrium.

INTRODUCTION

The Roos–Nash equilibrium [1–3], being the “strongest” of all known symmetric equilibria in the theory of noncooperative games, is of greatest interest for applications. However, it exists also in mixed strategies (unfortunately, only in rare cases [4–13]) in noncooperative program differential games and problems of acceptance or rejection of proposals, and even a passage to equilibrium with an accuracy of ε, in essence, does not expand the class of games having this solution. At the same time, the symmetric active equilibria [7, 14] (for brevity’s sake, called hereafter the A-equilibria), which in the ε-approximation exist in any problems, are known in the theory of problems of acceptance or rejection of a proposal, in which application of the Roos–Nash equilibrium concept is more correct than in noncooperative games. Such active equilibria can be applied both to the problems of acceptance of proposals and to noncooperative games, generally speaking, not in a smaller measure than the Roos–Nash equilibrium, which is their special case.

The question may arise whether a symmetric equilibrium exists, “weaker” than the Roos–Nash equilibrium and similar to it in content but “stronger” than the A-equilibrium. It is shown in the present paper that an analog of the Roos–Nash equilibrium existing in a wider class of game problems than the mentioned latter equilibrium can be determined on a set of A-equilibria in noncooperative games and problems of acceptance or rejection of proposals. This analog can be used not only in the problems of acceptance or rejection of proposals but also in classical noncooperative games.

Note that it is not the concepts of solutions of game problems applicable to all cases that are being considered in the theory of active equilibria, but only the possible concepts of stable situations used for construction of the theory of solution of problems, where the players participate not cooperating with each other and having different preferences. Numerous examples [7–14] confirm the expediency of application of active equilibria in search of solutions of not only the problems of “acceptance or rejection of a proposal” (formally, active equilibria were constructed for them), but also in search of stable states in noncooperative games.

It is proved that the new equilibrium introduced occupies a place in the hierarchical equilibria chain between the A-equilibrium and the B-equilibrium introduced in the theory of active equilibria [7, 14], followed by the Roos–Nash equilibrium. An example of a differential game is presented in which there is no Roos–Nash equilibrium in pure strategies but the new equilibrium proposed takes place.

---

1This study was supported by the Russian Fund for Basic Research, Projects Nos. 97-01-00123 and 97-01-00962.
FORMULATION OF THE PROBLEM

Let us consider the noncooperative differential program games of acceptance or rejection of proposals in mixed strategies. To simplify the presentation, especially due to consideration of the symmetric B-equilibrium, we will restrict the number of participants by two. However, all the results obtained are true for problems with any finite number of participants. Let the $i$th player, $i=1,2$, selecting the mixed strategy $q_i(u_i,t)$, tend to obtain a maximum of the functional

$$J_i(q) = \int_T dt \int_{W(t)} f_0(u,x,t) \, dq_1 \, dq_2, \quad i=1,2,$$

under the constraints

$$\dot{x} = \int_{W(t)} f(u,x,t) \, dq_1 \, dq_2,$$

$$x_j(t_0) = x_j^0, \quad j=1,n, \quad x_k(t_1) = x_k^1, \quad k \in K \subseteq 1,n,$$

$$\forall (u_1,u_2,t) \in W \subset E_1 \times E_2 \times T.$$ 

Let $x = (x_1, \ldots, x_n) \in E^n$; $E_i$, $i=1,2$, be finite-dimensional spaces, $W$ be a compact set in $E_1 \times E_2 \times T$; $W(t)$ be a section of the set $W$ at the moment $t \in T = [t_0, t_1]$; $K$ be a subset of the set of indices $\{1, n\}$; $U_1 = \text{Pr}_{E_1} W$ be a projection of the set $W$ onto $E_1$; $E_2 \triangleq E_1 \times E_2$; $Q_i$ be the set of mixed strategies $q_i(u_i,t)$ of the $i$th participant in problem (1), (2) with the initial condition $x(t_0) = x^0$ and with substitution of the set $W$ by some compact set $U_1 \times U_2$ (according to Theorems 4.2.1 and 4.2.6 from [15], the set $Q_i$ is a convex compact set in a *-weak topology of space $L^*_i(T, C(U_i)))$; $P_{q_i}(t)$ be the carrier of the measure $q_i(\cdot,t)$ at the moment $t \in T$, i.e., such a least closed set in $U_i$ whose complement has zero $q_i(\cdot,t)$-measure.

**Assumptions 1.** Let $T = [t_0, t_1]$ be a bounded fixed interval of the real axis $E^1$, the set $W$ be a compact set in $E_1 \times E_2 \times T$; $W(t)$ be a section of the set $W$ at the moment $t \in T = [t_0, t_1]$; $K$ be a subset of the set of indices $\{1, n\}$; $U_i = \text{Pr}_{E_i} W$ be a projection of the set $W$ onto $E_i$; $E \triangleq E_1 \times E_2$; $Q_i$ be the set of mixed strategies $q_i(u_i,t)$ of the $i$th participant in problem (1), (2) with the initial condition $x(t_0) = x^0$ and with substitution of the set $W$ by some compact set $U_1 \times U_2$ (according to Theorems 4.2.1 and 4.2.6 from [15], the set $Q_i$ is a convex compact set in a *-weak topology of space $L^*_i(T, C(U_i)))$; $P_{q_i}(t)$ be the carrier of the measure $q_i(\cdot,t)$ at the moment $t \in T$, i.e., such a least closed set in $U_i$ whose complement has zero $q_i(\cdot,t)$-measure.

**Assumptions 1.** Let $T = [t_0, t_1]$ be a bounded fixed interval of the real axis $E^1$, the set $W$ be a compact set in $E_1 \times E_2 \times T$; the mapping $f = (f_0, f_1, f_2, \ldots, f_n)^T: U \times E^n \times T \to E^{n+2}$ be such that the function $f(u, x, \cdot)$ is measurable (in the Lebesgue sense) for all $u \in U, x \in E^n$, and the function $f(\cdot, x, \cdot)$ for each $t \in T$ be continuously differentiable; let the function $\hat{f}$ be majorized on $T$ by the function $s(t)(|x|+1)$, where $s(t)$ is some integrable function: $x(t): T \to E^n$ be an absolutely continuous function satisfying Eq. (2). Moreover, the function $\hat{f}$ satisfies with the integrable function $b(t)$ the Lipschitz condition

$$|\hat{f}(u, x, t) - \hat{f}(u, \bar{x}, t)| \leq b(t)|x - \bar{x}|$$

for all $u \in U$, $x, \bar{x} \in E^n$, $t \in T$. 

Let $G$ be a subset of the compact set $Q_1 \times Q_2$, formed only by such strategies $q_i$, which allow us to provide constraints (3), (4), with constraints (3) introducing into the problem (1)–(4) an implicit dependence between the strategies, and constraint (4) introducing an explicit dependence due to the fact that at almost (in the sense of the Lebesgue measure) each moment $t \in T$, the measures $q_i(\cdot,t)$ can be selected only so that a direct product $P_{q_1q_2}(t)$ of their carriers $P_{q_1}(t)$ and $P_{q_2}(t)$ is brought into the set $W(t)$. Only such measures are considered admissible. Thus, the set $G$ is the set of only such pairs of measures $q_1(\cdot,t)$ and $q_2(\cdot,t)$, the product of whose carriers at each moment $t \in T$ lies in $W(t)$, i.e., $P_{q_1}(t) \times P_{q_2}(t) \subset W(t)$. $\text{Pr}_{Q_i} G$ is the projection of the set $G$ onto the space $Q_i$.

**Lemma 1.** Let problem (1)–(4), in which $G$ is not empty, satisfy Assumptions 1. Then:
1) each nonempty section $G(q_i), i=1,2$, is compact;
2) for any sequence $q_i^k \in G(q_k), k \neq i$, converging to $q_i \in G(q_i)$ in a topology induced in $G(q_k)$ by the *-weak topology of the space $L^*_i(T, C(U_i)))$, the sequence $x^k(t)$ as a solution of system (2)–(4) converges to the solution $x(t)$ in a uniform topology and the mapping $x: G(q_k) \to X \subset C(T, E^n)$, where $X$ is the set of trajectories of system (2)–(4), is continuous;