THE PROBLEM OF SYNTHESIS OF RELIABLE NETWORKS

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The problem of determination of a throughput of network edges on an undirected graph with a minimal total network cost is considered in the paper, provided that, if any edge is removed from the graph, ways exist along which a single flow goes from the source to the sink in the obtained network.

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In problems arising in the design of networks of various application, usually the problem is considered of synthesis of networks realizing passage of a statistical flow from a producer to a consumer with a minimal total cost, without considering the reliability of network functioning in the case of failure of its individual elements. The problem of synthesis of reliable networks was considered in papers [1, 2] in the probabilistic aspect with regard to their reliability.

The reliability of functioning of product-conducting networks can be associated with the possibility to transport the required flow from a producer to a consumer in the case of damage of one of the network sections. The probability of a simultaneous damage in two sections of the network is assumed to be negligibly small.

Consideration of single failures in solving synthesis problem provides a high reliability of functioning of the network being designed in actual practice.

In ordinary situations, a damage in product-conducting networks can result from corrosion of materials of the communication line structure. Since the frequency and the theoretically calculated probability of development of a damage often differ, it is better to use the degree of destruction of materials of the structure (strength) of the communication line in a definite period instead of the probability of development of a damage in a definite period. In this case, the weight of elements (arcs) of a network can be estimated by the quantity \( \mu r \), where \( \mu \) is the strength of the proposed material for creation of communication and \( r \) is the real length of the section.

Let us assume that an unoriented network \( G = (V, E) \) is given with the set of nodes \( V \), set of edges \( E \), and the nodes \( s, r \in V \), are fixed as the source and the sink, respectively. For each edge \( e \in E \), its weight \( c_e = \mu_r e \geq 0 \) is given. Let \( x_e \) be the unknown throughput of the edge \( e \) for all \( e \in E \). The costs of creation of the edge \( e \) with the throughput \( x_e \) is equal to \( c_e x_e \) for all \( e \in E \). It is required to determine throughputs \( x_e \geq 0 \) of edges \( e \in E \) provided that with removal of only one arbitrary edge of the network \( G \), passage of a given flow from the source \( s \) to the sink \( r \) can be provided in such a way that the cost of the network as the sum of the costs of edges is minimal.

Let us introduce auxiliary variables \( x_{ij}(e) \) designating the flow on the edges \( (i, j) \in E \) directed from the node \( i \) to the node \( j \) when an edge \( e \neq (i, j) \) is removed from the network \( G \) for all \( e \in E \). In this notation the mathematical model has the following form: find

\[
\min f(x) = \min \sum_{(i, j) \in E} c_{ij} x_{ij}
\]

under the constraints

\[
\sum_{j \in \delta(i, e)} x_{ji}(e) - \sum_{j \in \gamma(i, e)} x_{ij}(e) = \begin{cases} 
-u & \text{for } i = s, \\
0 & \text{for } i \neq s, r, i \in V, e \in E, \\
v & \text{for } i = r,
\end{cases}
\]
Here, $\delta(i, e)$ is the set of initial nodes $j$ of the edges $(j, i)$ and $\gamma(i, e)$ is the set of terminal nodes of the edges $j(i, j)$ in the network obtained after removal of the edge $e$ of $G$.

We can assume in problem (1)-(4) without loss of generality that the preset amount of flow from $s$ and $r$ is equal to unity. Thus we put $v = 1$. Generally, the graph $G$ can be both oriented and unoriented. This problem with additional conditions

$$x_{ij} = 0 \forall (i, j) \in E$$

is equivalent to the problem of finding a flow with a minimal cost [3] with the value equal to 2 with the unit throughput of all edges of $E$.

Problem (1)-(4) without additional conditions (5) arises as a subproblem which should be solved for each iteration of the branch-and-bound algorithm in calculating the lower bound for the problem of design of reliable networks of different application.

As distinct from problem (1)-(5), the process of finding the solution of problem (1)-(4) is more labor-consuming. To explain what has been said, we will consider the algorithm of solution of problem (1)-(4) when $G$ is the network shown in Fig. 1.

In this figure (and further), the number inside a circle is the number of the node, and the numbers adjacent to edges are their weights.

The network shown in Fig. 1 is very simple and a failure of one edge of any path is equivalent to the failure of all edges of this path. It is apparent that the throughputs of edges of one path are equal and we will call this value the throughput of the respective path.

Let $y_i$ be the unknown throughput of the path $i$, and $d_i$ be its length (the sum of weights of edges of this path).

In the network in Fig. 1 $d_1 = 2, d_2 = 3, d_3 = 4, d_4 = 5$, and problem (1)-(4) is equivalent to the following problem: find $\min 2y_1 + 3y_2 + 4y_3 + 5y_4$ under the constraints

$$y_2 + y_3 + y_4 \geq 1,$$

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$$y_i \geq 0, \ i = 1, 2, 3, 4.$$  

Putting $y_1 = 1, y_2 = 1, y_3 = 0, y_4 = 0$, i.e., putting the throughputs of the first and second shortest paths equal to unity, we obtain an optimal and admissible solution of problems (1)-(5) and (1)-(4), respectively. The admissible solution of problem (1)-(4) is nonoptimal. We will determine another admissible solution by putting $y_1 = \frac{1}{2}, y_2 = \frac{1}{2}, y_3 = \frac{1}{2}, y_4 = 0$. Since $d_1 + d_2 > d_3$, this solution is better than the preceding one. Checking for the “optimality” conditions

$$\frac{d_1}{2} + \frac{d_2}{2} + \frac{d_3}{2} < d_4,$$

we obtain that the last admissible solution of problem (1)-(4) is optimal.