AN ALGORITHM FOR CONSTRUCTING A VARIETY OF ARBITRARY FINITE DIMENSION

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The dimension of a variety $V$ of algebras is the greatest length of a basis (i.e., of an independent generating set) for an SC-theory $SC(V)$, consisting of strong Mal'tsev conditions satisfied in $V$. The variety $V$ is assumed infinite-dimensional if the lengths of the bases in $SC(V)$ are not bounded. A simple algorithm is found for constructing a variety of any finite dimension $r \geq 1$.

Using the sieve of Eratosthenes, $r$ distinct primes $p_1, p_2, \ldots, p_r$ are written and their product $n = p_1 p_2 \ldots p_r$ is computed. The variety $G_n$ of algebras $(A, f)$ with one $n$-ary operation satisfying the identity

$$f(z_1, z_2, \ldots, z_n) = f(z_2, \ldots, z_n, z_1)$$

has, then, dimension $r$.

INTRODUCTION

In [1], the dimension $\dim (V)$ of a variety $V$ of algebras was defined to be the greatest length of a basis (i.e., of an independent generating set) for an SC-theory $SC(V)$ of $V$. The dimension is thought of as infinite if the lengths of the bases in $SC(V)$ are unbounded. Thus, $\dim (V)$ is also the dimension of the $SC$-theory $SC(V)$.

In [2], it was shown that a finitely based variety $V$ may be infinite-dimensional. An example is any Cantor variety $C_{m,n}$ which has $m (\geq 1)$ $n$-ary $\{g_1, \ldots, g_m\}$ and $n (\geq m)$ $m$-ary $\{f_1, \ldots, f_n\}$ operations and is defined by the following identities:

$$f_i(g_1(z_1, \ldots, z_n), \ldots, g_m(z_1, \ldots, z_n)) = z_i, \quad i = 1, \ldots, n,$$

$$g_j(f_1(z_1, \ldots, z_m), \ldots, f_n(z_1, \ldots, z_m)) = z_j, \quad j = 1, \ldots, m.$$

By contrast, every Post variety $P_n$ generated by a finite primal algebra of order $n \geq 2$ is finite-dimensional. In [2], in particular, it was shown that the dimension of a variety of Boolean algebras $(B, +, \cdot, ', 0, 1)$ is at most 4.

The goal of the present article is to present an algorithm for constructing a finitely presented variety (by which is meant one that is finitely based and of finite signature) of arbitrary finite dimension $r \geq 1$.

We recall the basic definitions needed. The class $[V]$ of varieties equivalent, with respect to interpretability, to a given variety $V$ is commonly called an interpretability type (IT for short). If we put $[V] \leq [V']$ whenever $V$ is interpretable in $V'$ we obtain a lattice $L_{\text{int}}^\text{f}$ in which the finite types (i.e., IT's of finitely presented varieties) form a sublattice $L_{\text{int}}^f$. For a finitely presented variety $V$, its dimension can be defined to be the greatest number $r$ for which there exist finitely presented varieties $V_1, V_2, \ldots, V_r$ such that $L_{\text{int}}^f$ satisfies the following two conditions:

1. $[V_1] \cup [V_2] \cup \ldots \cup [V_r] = [V]$;
2. $\bigvee_{i \neq k} [V_i] < [V]$ for every $k = 1, 2, \ldots, r$.
For any variety \( W \in [V] \), \( \dim(W) = \dim(V) \), and so we also call \( \dim(V) \) the dimension of a type \([V]\).

The collection of all strong Mal'tsev conditions satisfied in a variety \( V \) is referred to as an \textit{SC-theory} of \( V \). A \textit{strong Mal'tsev condition} is a formula of the form

\[
M: (\exists f_1) \ldots (\exists f_k)(\forall z_1) \ldots (\forall z_n) \bigwedge_{i=1}^m t_i = t'_i,
\]

where \( t_i \) and \( t'_i \) are terms in the function symbols \( f_1, \ldots, f_k \) and in the individual variables \( z_1, \ldots, z_n \). We say that \( M \) is satisfied in a given variety \( V \) if there exist terms \( \tilde{f}_1, \ldots, \tilde{f}_k \) in the language of \( V \) such that \( V \models (\forall z_1) \ldots (\forall z_n) \tilde{M} \), where \( \tilde{M} \) is obtained from \( M \) by replacing each function symbol \( f_j \) by a term \( \tilde{f}_j \).

The equalities \( t_i = t'_i \), forming a strong condition \( M \), specify a finitely presented variety \( W_M \), and the fact that \( M \) is satisfied in \( V \) is equivalent to \( W_M \) being interpretable in \( V \). In turn, for each finitely presented variety \( V \), there is a strong Mal'tsev condition \( M \), for which \( W_M = V \).

If \( M_1, \ldots, M_r \) are the strong Mal'tsev conditions corresponding to finitely presented varieties \( V_1, \ldots, V_r \), then condition (1) in the definition of a dimension means that \( \{ M_1, \ldots, M_r \} \) is a generating set for \( SC(V) \); condition (2) shows that the set \( \{ M_1, \ldots, M_r \} \) is independent, that is, for every \( M_k \), there exists a variety \( U_k \) in which all \( M_i, i \neq k \), are satisfied but \( M_k \) is not. As \( U_k \) we can take the coproduct variety

\[
V_1 \coprod \cdots \coprod V_{k-1} \coprod V_{k+1} \coprod \cdots \coprod V_r,
\]

which is a variety presented by disjoint unions \( \bigcup_{i \neq k} \Omega(V_i) \) (of signatures) and \( \bigcup_{i \neq k} \text{Id}(V_i) \) (of defining identities).

1. THE GENERALIZED GUMM THEOREM

Gumm in [3] proved that the strong Mal'tsev condition

\[
(\exists f) f(x, y) = f(y, z)
\]

is satisfied in a certain variety \( V \) iff each involution \( \varphi \) of any algebra \( A \) in \( V \) has a fixed point.

Garcia and Taylor in [4] noticed that Gumm's result will also be valid if stated in a more general form. The route that we follow up to prove the Gumm theorem is as follows.

For every variety \( V \), the following three conditions are equivalent:

(a) the strong Mal'tsev condition

\[
M_n: (\exists f) f(x_1, x_2, \ldots, x_n) = f(x_2, \ldots, x_n, x_1) \quad (n \geq 2)
\]

is satisfied in \( V \);

(b) every automorphism \( \varphi \) of any algebra \( A \) in \( V \), whose \( n \)th degree has a fixed point, has a fixed point itself;

(c) every automorphism \( \varphi \) of order \( n \) of any algebra \( A \) in \( V \) has a fixed point.

In fact, suppose that there exists a term \( \tilde{f} \) in the language of \( V \) such that the identity \( \tilde{f}(x_1, x_2, \ldots, x_n) = \tilde{f}(x_2, \ldots, x_n, x_1) \) is true in \( V \), and \( \varphi \) is an automorphism of an arbitrary algebra \( A \) of \( V \) satisfying the condition

\[
(\exists z \in A) z^{\varphi^n} = z.
\]

For such \( z \), then, the element \( a = \tilde{f}(z, z^{\varphi}, \ldots, z^{\varphi^{n-1}}) \) is fixed with respect to \( \varphi \) since

\[
a^{\varphi} = \tilde{f}(z^{\varphi}, z^{\varphi^2}, \ldots, z^{\varphi^{n-1}}, z) = \tilde{f}(z, z^{\varphi}, \ldots, z^{\varphi^{n-1}}) = a.
\]