GROUPS CONTAINING AN ELEMENT THAT PERMUTES WITH A FINITE NUMBER OF ITS CONJUGATES

V. E. Kislyakov*

We study a group $G$ containing an element $g$ such that $C_G(g) \cap g^G$ is finite. The nonoriented graph $\Gamma$ is defined as follows. The vertex set of $\Gamma$ is the conjugacy class $g^G$. Vertices $x$ and $y$ of the graph $G$ are bridged by an edge iff $x \neq y$ and $xy = yx$. Let $\Gamma_0$ be some connected component of $G$. On a condition that any two vertices of $\Gamma_0$ generate a nilpotent group, it is proved that a subgroup generated by the vertex set of $\Gamma_0$ is locally nilpotent.

INTRODUCTION

An arbitrary group $G$, as is known, can be represented as an automorphism group of the graph $\Gamma$ defined as follows. Let $a$ be an element of $G$. The vertex set $V(G)$ of $\Gamma$ is the conjugacy class $a^G$. The edge set consists of unordered vertex pairs $\{z, y\}$ such that $z \neq y$ and $zy = yz$. Thus, $\Gamma$ is a nonoriented graph without loops and multiple edges. The group $G$ acts on the set $V(\Gamma)$ by conjugation and is a vertex-transitive group of automorphisms of $\Gamma$.

We use this construction to study the situation in which the element $a$ of $G$ permutes with a finite number of its conjugates, that is, the set $C_G(a) \cap a^G$ is finite. Note that the situation that we are envisaging can well be critical, as is, for instance, one encountered in constructing infinite Abelian subgroups in a group.

Thus, if $|C_G(a) \cap a^G| < \infty$, then $\Gamma$ is a locally finite graph. In what follows, it might be convenient to refer to elements of the conjugacy class $a^G$ as vertices of the graph $\Gamma$. In [1], the following question was posed: What is the condition under which the connected components of a locally finite graph $\Gamma$ are finite? Let $\Gamma_0$ be some connected component of $\Gamma$ and $V(\Gamma_0)$ its vertex set. Since all connected components of $\Gamma$ are conjugate in $G$, one connected component being finite implies that all connected components of $\Gamma$, too, are finite. In certain cases, the answer to the question posed above can be given by studying into the structure of a subgroup $V(\Gamma_0)$.

In the present article, we deal with a subgroup generated by the vertex set of a connected component in the locally finite graph $\Gamma$, subject to some additional finiteness conditions.

Let $H = \langle V(\Gamma_0) \rangle$. The subgroup $H$ is characterized by a number of interesting properties, which are independent of the structure of $G$. Before embarking on them, we recall certain of the concepts. Subgroups $X$ and $Y$ are called commensurable if the indices $|X : X \cap Y|$ and $|Y : X \cap Y|$ are finite. A subgroup $X$ is said to be FC-embedded in $G$ if, for any element $g$ in $G$, the index $|X : C_X(g)|$ is finite. The following holds:

Statement 1. (1) if $z \in V(\Gamma_0)$ then $|C_H(z) \cap z^H| < \infty$;

*Supported by the RF State Committee of Higher Education.

Proposition 2. Let $G$ be a group, $\Gamma$ a locally finite graph, and $\Gamma_0$ some connected component of $\Gamma$. Then $V(\Gamma_0)$ is a finite set if and only if $H$ is an almost central group.

We pass to state the main result of the present article.

**Theorem 3.** Let $G$ be a group, $\Gamma$ a locally finite graph, $\Gamma_0$ some connected component of $\Gamma$, and $H = \langle V(\Gamma_0) \rangle$. If any two vertices of $\Gamma_0$ generate a nilpotent group, then $H$ is a locally nilpotent group.

Theorem 3 implies the following:

**Corollary 4.** Let $G$ be a group, $\Gamma$ a locally finite graph, $\Gamma_0$ its connected component, and $H = \langle V(\Gamma_0) \rangle$. If any two vertices of $\Gamma_0$ generate a finite nilpotent group, then $H$ is a periodic locally nilpotent group.

**Proof.** By assumption, all elements in the set $V(\Gamma_0)$ have finite order. By Theorem 3, the subgroup $H$ is locally nilpotent. Hence $V(\Gamma_0)$ is contained in the periodic part $T(H)$ of $H$. Consequently, $H = T(H)$, proving the corollary.

Using Corollary 4, it is not hard to obtain the next:

**Corollary 5.** Let $G$ be a group, $\Gamma$ a locally finite graph, $\Gamma_0$ its connected component, and $H = \langle V(\Gamma_0) \rangle$. If any two vertices of $\Gamma_0$ generate a finite $p$-group, then $H$ is a locally finite $p$-group.

The results given above indicate that Theorem 3 still does not allow us to make the conclusion as to whether the connected components of $\Gamma$ are finite. We are unaware of any example denying this situation, that is, the fact of there being a group for which there exists an infinite connected locally finite graph $\Gamma$ any two vertices of which would generate a nilpotent group. Our nearest goal is to refine the structure of $H$. Here we distinguish three cases — where $H$ is torsion free, mixed, or periodic. The situation in which $H$ is torsion free can be thought of as being clearly understood. The following holds:

**Proposition 6.** Let $G$ be a group, $\Gamma$ a locally finite graph, $\Gamma_0$ some connected component of $\Gamma$, and $H = \langle V(\Gamma_0) \rangle$. If any two vertices of $\Gamma_0$ generate a nilpotent group, and the subgroup $H$ is torsion free, then $H$ is a finitely generated Abelian group.

By Proposition 2, it follows that $V(\Gamma_0)$ is a finite set, and consequently the connected components of $\Gamma$ are finite.

The next proposition is of interest for case where $H$ is a mixed group.

**Proposition 7.** Let $G$ be a group, $\Gamma$ a locally finite graph, $\Gamma_0$ some connected component of $\Gamma$, and $H = \langle V(\Gamma_0) \rangle$. If any two vertices of $\Gamma_0$ generate a nilpotent group, and the subgroup $H$ is finitely generated, then $H$ is almost central.

At this point, we make some comments on the content of the article. In Secs. 1 and 2, we give auxiliary statements. The results stated in the Introduction are proved in Sec. 3. We omit the proof of Statement 1 by reason of the fact that clause (1) is obvious, and (2)-(6) can be easily derived, for instance, from [1, Lemmas 1.1 and 1.4].